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Best Proximity Points for Some Classes of Nonlinear Operators

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Abstract of the PhD Thesis

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Bucharest, 2020

Key words and phrases: fixed point; metric space; gauge space; best proximity point; implicit simulation function; χ-proximal contraction; U-property; directed graph; strictly α-proximal admissible mapping; F-α-proximal contraction; approximately compact set; simulation function; Geraghty's proximal contraction; R-function; R-proximal contraction; implicit generalized proximal contraction; nonself mapping; CQ algorithm.

2010 Mathematics Subject Classification: 47H10; 54H25

Contents

List of author's published papers

1. Houmani, H, Ţurcanu, T: CQ-type algorithm for reckoning best proximity points of EP-operators, Symmetry-Basel, 12(2020), No. 1, Art. No. 4.

2. Ali, MU, Farheen, M, Houmani, H: On a study of best proximity points for R-proximal contractions in gauge spaces. U. Politeh. Buch. Ser. A 82(2020), No. 1, 151-160.

3. Ali, MU, Houmani, H, Kamran, T: New type of proximal contractions via implicit simulation functions. J. Nonlinear Convex Anal. 20(2019), No. 3, 435-445.

4. Ali, MU, Farheen, M, Houmani, H: Best proximity point theorems for implicit proximal contractions on gauge spaces. J. Math. Anal. 8(2017), No. 6, 180-188.

5. Ali, MU, Fahimuddin, Kamran, T, Houmani, H: Best proximity points of F-proximal contractions under the influence of an alpha-function. U. Politeh. Buch. Ser. A 79(2017), No. 4, 3-18.

Conference talk:

1. Houmani, H: New type of proximal contractions via implicit simulation functions, Second Romanian Itinerant Seminar on Mathematical Analysis and Its Applications, Constanta, May 10-12, 2019.

Abstract

Best proximity theory has developed extensively as a response to a question which has its roots in generalizing fixed point problems. If (X, d) is a metric space, and A, B are two nonempty subsets of X, let $T\colon A \to B$ be a mapping. In this case, the set of the fixed points of the mapping T may be void, or even worse, the problem might not have a meaning (if the two sets involved, A and B, have their intersection void). Naturally, in such situations, the problem transforms into finding a point from the set A so that the distance from it to its image through T is smallest in some sense. In this respect, Fan [14] studied the situation in which A is endowed with some adequate properties of convexity and compactness, $B = X$, and the operator T is continuous, and he proved that there exists a point x in the set A for which the distance $d(x, Tx)$ is precisely the distance from Tx to A. From that point onwards, such global minimization problems became the object of study for many scientists, who designed various techniques in this regard. One such direction is searching for a best proximity point x of a mapping T, which is a point for which the distance from it to its image Tx equals the distance between the sets A and B. The development of this research area has three main directions. The first refers to obtaining generalized contractive conditions which ensure the existence of best proximity points; the second has in view the use of suitable generalized underlying metric spaces; the third is the numerical reckoning of best proximity points.

The PhD. thesis "Best Proximity Points for Some Classes of Nonlinear Operators" contains our contribution to the best proximity theory, and presents results obtained in the study of each of the three directions previously mentioned. Following the fixed point approach, in this thesis we formulated best proximity results for some classes of proximal operators, which are global optimization results associated with distances between two sets involved in the definition of the considered operator, both for single valued and also for multivalued operators. The settings chosen are the classic metric spaces, and gauge spaces. The reckoning of best proximity points is illustrated by a CQ-type algorithm related with (EP)-operators.

The first part in this thesis consists of Chapter 1 and Chapter 2, and aims to establish some best proximity point results in metric spaces, by using the approach where the problem is treated as that of finding a global optimal approximate solution to the fixed point equation for adequate proximal contraction mappings. Chapter 1 is dedicated to single valued operators, and some best proximity point theorems are stated and proved; also some classic results are obtained as corollaries of our results. In Chapter 2, the same problem of global optimization is studied, but for multivalued operators. This part generalizes some results obtained by Sadiq Basha and Shahzad [8], Jleli and Samet [22], Turinici [31], or Wardowski [32]. In the second part of this Thesis (Chapter 3) we have changed the framework, and have used gauge spaces to study the problem of best proximity for nonself mappings satisfying proximal contractive type conditions based on auxiliary functions, such as the R-functions, the simulation functions, or the Geraghty

functions. Implicit generalized proximal contraction mappings are used in the last part of the chapter in order to study the best proximity point problem. The research is a natural continuation of some ideas of Cherichi *et al.* [11], Hierro and Shahzad [19], Khojasteh *et al.* [23], or Mongkolkeha *et al.* [26]. Numerical reckoning of best proximity points is featured in the third part of this thesis (Chapter 4), by the construction of a CQ-type algorithm which generates sequences which converge strongly to best proximity points of mappings satisfying a new contractive condition (the (EP)-condition). Here we develop further some ideas of Gabeleh [16], García-Falset *et al.* [17], Nakajo and Takahashi [27] (see also Takahashi [28]), and Thakur *et al.* [29].

In Chapter 1, entitled **Proximal contractions via implicit simulation functions**, we consider a problem of global optimization, more precisely the problem of the minimum distance between two subsets A and B of a nonempty set X. By using the (c)-comparison functions, a class of functions is introduced, which are continuous, monotone with respect with some of its variables, and fulfill some additional hypotheses defined by means of inequalities. More accurately, we need functions $\kappa\colon (\mathbb{R}^+)^4 \to \mathbb{R}^+$ ($\mathbb{R}^+ = [0,\infty)$) satisfying the following conditions:

(i) κ is continuous, and nondecreasing as a function of the first variable only and of the fourth variable, that is $\kappa(\cdot, u, v, w)$ and $\kappa(u, v, w, \cdot)$ are nondecreasing, for any fixed u, v, $w \in \mathbb{R}^+$;

(ii) If $p \geq q$ and $p \leq \kappa(p, q, p, p)$, then $p = 0$;

(iii) If $p < q$ and $p \leq \kappa(q, q, p, q)$, then $p \leq \psi(q)$.

Such kind of functions allow us to introduce the implicit simulation functions, as follows.

Definition 0.1 ([20], Definition 1.4)**.** A function $\chi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called an implicit simulation function with respect to κ, if the following axioms hold:

(i) $\chi(cp, \kappa(q, r, s, t)) \leq \kappa(q, r, s, t) - cp$, for any real number $c \geq 0$, and $p \geq 0$;

(ii) $\chi\left(l, \kappa\left(0, 0, l, \frac{l}{2}\right)\right) \geq a$ if and only if $\kappa\left(0, 0, l, \frac{l}{2}\right) - l \geq a$, for any real number a;

(iii) $\chi\left(l, \kappa\left(0, 0, l, \frac{l}{2}\right)\right) \geq 0$ implies $l = 0$.

This notion is the main tool in the definition of our first implicit generalized contraction, with respect to functions κ with the properties previously mentioned.

Definition 0.2 ([20], Definition 1.5)**.** Consider (X, d) a metric space, A and B two nonempty subsets of X, and $\alpha\colon A \times A \to [0, \infty)$ be a mapping. Let χ be an implicit simulation function with respect to κ. A mapping $T\colon A \to B$ is called a χ-proximal contraction of the first kind if, for each x_1, x_2, u_1, $u_2 \in A$, the equalities $d(u_1, Tx_1) = d(A, B) = d(u_2, Tx_2)$ imply

$$\chi\Big(\alpha(x_1, x_2) d(u_1, u_2), \kappa\Big(d(x_1, x_2), d(x_1, u_1), d(x_2, u_2), \frac{d(x_2, u_1) + d(x_1, u_2)}{2}\Big)\Big) \geq 0.$$

This class of implicit contractions contains some classic contractive conditions; adequate choices of α, κ and χ have led us to this statement.

By imposing additional conditions eg the α-proximal admissibility of the considered operator, or an approximately compactness condition, we stated and proved the existence of a best proximity point of χ-proximal contractions of the first kind, by using methods specific to best proximity theory.

Theorem 0.1 ([20], Theorem 1.1)**.** *Let (X, d) be a complete metric space, and A and B be nonempty and closed subsets of X. Let $T\colon A \to B$ be a χ-proximal contraction of the first kind, and $\alpha\colon A \times A \to \mathbb{R}^+$ be a mapping. Suppose the following conditions hold:*

(i) *T is α-proximal admissible;*

(ii) *there exist x_0, $x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$, and $\alpha(x_0, x_1) \geq 1$;*

(iii) *$T(A_0) \subseteq B_0$;*

(iv) *B is approximately compact with respect to A;*

(v) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, for each $n \in \mathbb{N}$, and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$, for each $n \in \mathbb{N}$.*

Then T has a best proximity point.

By considering certain functions κ, and χ, some classical best proximity results can be obtained as consequences of this theorem.

Theorem 0.1 does not ensure the uniqueness of a best proximity point, and the thesis provides an example to prove that there are not enough hypotheses to allow us to draw this conclusion. In order to formulate a uniqueness result, some additional conditions have been introduced, one referring to a monotone type condition with respect to the third variable of the function κ, and the other is the so-called U property, namely if (X, d) is a metric space, A and B are subsets of X, and $T\colon A \to B$ is a nonself mapping, it is said that T is endowed with the U property with respect to $\alpha\colon A \times A \to [0, \infty)$, if for each $x \neq y$ best proximity points of T, we have $\alpha(x, y) \geq 1$.

Theorem 0.2 ([20], Theorem 1.2)**.** *Suppose the hypotheses from Theorem* 0.1 *are satisfied, and additionally, that T is endowed with the U property, and $\kappa(u, v, \cdot, w)$ is also nondecreasing, for any taken u, v, $w \in \mathbb{R}^+$. Then, the best proximity point of T is unique.*

By changing some parts of the inequality which mainly defines the χ-proximal contractions of the first type, we obtained another kind of χ-proximal contractions.

Definition 0.3 ([20], Definition 1.7)**.** Let (X, d) be a metric space, A and B be two nonempty subsets of (X, d), and $\alpha\colon A \times A \to [0, \infty)$ be a mapping. Let χ be an implicit simulation function with respect to κ. A mapping $T\colon A \to B$ is called a χ-proximal contraction of the second kind if, for each $x_1, x_2, u_1, u_2 \in A$, the equalities $d(u_1, Tx_1) = d(A, B) = d(u_2, Tx_2)$ imply

$$\chi\Big(\alpha(x_1, x_2)d(Tu_1, Tu_2), \kappa\Big(d(Tx_1, Tx_2), d(Tx_1, Tu_1), d(Tx_2, Tu_2), \frac{d(Tx_2, Tu_1) + d(Tx_1, Tu_2)}{2}\Big)\Big) \geq 0.$$

This contractive condition, jointly with some hypotheses almost similar to those in Theorem 0.1, lead to an existence result regarding best proximity points.

Theorem 0.3 ([20], Theorem 1.3)**.** *Let (X, d) be a metric space, and A and B be nonempty and closed subsets of X. Let $T\colon A \to B$ be a χ-proximal contraction of the second kind. Assume the following conditions hold:*
(i) *T is α-proximal admissible;*
(ii) *there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$, and $\alpha(x_0, x_1) \geq 1$;*
(iii) *$T(A_0) \subseteq B_0$;*
(iv) *A is approximately compact with respect to B;*
(v) *T is continuous.*
Then T has a best proximity point.

The second section of this chapter is dedicated to fixed point results obtained as applications of the results in the first section of the chapter, as follows.

Theorem 0.4 ([20], Theorem 1.4)**.** *Let (X, d) be a complete metric space and let $T\colon X \to X$ be a mapping such that*

$$\chi\Big(\alpha(x_1, x_2)d(Tx_1, Tx_2), \kappa\Big(d(x_1, x_2), d(x_1, Tx_1), d(x_2, Tx_2), \frac{d(x_2, Tx_1) + d(x_1, Tx_2)}{2}\Big)\Big) \geq 0,$$

for all $x_1, x_2 \in X$, where χ is an implicit simulation function with respect to κ and $\alpha\colon X \times X \to [0, \infty)$ be a mapping. Further, assume that the following conditions hold:
(i) *T is α-admissible, that is for $x, y \in X$ with $\alpha(x, y) \geq 1$, we have $\alpha(Tx, Ty) \geq 1$;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$, for each $n \in \mathbb{N}$.*
Then T has a fixed point.

Theorem 0.5 ([20], Theorem 1.6)**.** *Let (X, d) be a complete metric space and let $T\colon X \to X$ be a mapping such that*

$$\chi\Big(\alpha(x_1, x_2)d(T^2x_1, T^2x_2), \kappa\Big(d(Tx_1, Tx_2), d(Tx_1, T^2x_1), d(Tx_2, T^2x_2), \frac{d(Tx_2, T^2x_1) + d(Tx_1, T^2x_2)}{2}\Big)\Big) \geq 0,$$

for all $x_1, x_2 \in X$, where χ is an implicit simulation function with respect to κ, and $\alpha\colon X \times X \to [0, \infty)$ be a mapping. Further, assume that the following conditions hold:
(i) *T is α-admissible, that is for $x, y \in X$ with $\alpha(x, y) \geq 1$, it follows $\alpha(Tx, Ty) \geq 1$;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *T is continuous.*
Then T has a fixed point.

As important corollaries which follow from these theorems, we mentioned, among others, the Banach principle, and the classic Kannan and Chatterjea results; moreover, considering that X is a nonempty set endowed with a directed graph $G = (V; E)$, where the set of its vertices is $V = X$, and the set of its edges contains all loops with no parallel edges, some fixed point results for metric spaces endowed with such graphs are obtained as consequences of the previous theorems. The outcome of this chapter comprises three definitions, six theorems, and thirteen corollaries.

Chapter 2, "F-proximal contractions with α-functions", has its starting point in the implicit contractive condition introduced by Wardowski [32], who studied fixed point properties related to the notion he had introduced. His result was generalized in Minak *et al.* [25], or Sgroi and Vetro [30]. Our first generalization is of Hardy-Rogers type, and is made by using mainly functions from the set $\mathfrak{F}$, which contains all functions $F\colon (0,\infty) \to \mathbb{R}$ which fulfill the next properties:

(F_1) The function F is strictly nondecreasing.

(F_2) For each sequence $\{\mathfrak{d}_n\}$ of positive real numbers we have $\lim_{n\to\infty} \mathfrak{d}_n = 0$ if and only if $\lim_{n\to\infty} F(\mathfrak{d}_n) = -\infty$.

(F_3) There exists $k \in (0,1)$ such that $\lim_{n\to\infty} \mathfrak{d}_\mathfrak{n}{}^k F(\mathfrak{d}_\mathfrak{n}) = 0$, whenever $\lim_{n\to\infty} \mathfrak{d}_n = 0$.

In the following, H is the generalized Hausdorff metric induced by a metric d.

Definition 0.4 ([20], Definition 2.4)**.** Let (X,d) be a metric space, A, B nonempty subsets of X. Let $F \in \mathfrak{F}$ and $\alpha\colon A\times A \to [0,\infty)$ be a function. A mapping $T\colon A \to CL(B)$ is a F-α-proximal contraction of Hardy-Rogers type if there exists $\tau > 0$ such that

$$\tau + F(\alpha(x,y)H(Tx,Ty)) \le F(N(x,y)),$$

for each $x,y \in A$, whenever $\min\{\alpha(x,y)H(Tx,Ty), N(x,y)\} > 0$, where

$$\begin{aligned} N(x,y) \;=\;& a_1 d(x,y) + a_2[d(x,Tx) - d(A,B)] + a_3[d(y,Ty) - d(A,B)] \\ & + a_4[d(x,Ty) - d(A,B)] + L[d(y,Tx) - d(A,B)], \end{aligned}$$

with $L \ge 0$ and a_1, a_2, a_3, $a_4 \ge 0$ satisfy $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \ne 1$.

By imposing some other conditions to a Hardy-Rogers type contraction, such as α-admissibility, the weak P-property, a continuity assumption of this mapping or, alternatively, a hypothesis defined by means of the function α, the existence of a best proximity point for such kind of generalized contractions was stated and proved.

Theorem 0.6 ([20], Theorem 2.4)**.** *Let (X,d) be a metric space, A and B be nonempty, closed subsets of X. Assume that $T\colon A \to CL(B)$ is a F-α-proximal contraction of Hardy-Rogers type satisfying the following conditions:*

(i) *$Tx \subseteq B_0$ for each $x \in A_0$ and (A,B) satisfies the weak P-property;*

(ii) *T is strictly α-proximal admissible;*

(iii) *there exist $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$\alpha(x_0,x_1) > 1 \quad \textit{and} \quad d(x_1,y_1) = d(A,B);$$

(iv) *T is continuous, or, for any sequence $\{x_n\} \subseteq A$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n,x) > 1$ for each $n \in \mathbb{N}$.*

Then T has a best proximity point.

Another generalization we had in view is of Ćirić type, and is based on functions from the family $\mathfrak{F}$, and auxiliary functions α.

Definition 0.5 ([20], Definition 2.5). Let (X, d) be a metric space, A, B nonempty subsets of X. Consider a continuous mapping F in $\mathfrak{F}$ and a function $\alpha\colon A \times A \to [0, \infty)$. A mapping $T\colon A \to CL(B)$ is called F-α-proximal contraction of Ćirić type if there exists $\tau > 0$ such that

$$\tau + F(\alpha(x, y)H(Tx, Ty)) \leq F(M(x, y)),$$

for each $x, y \in A$, whenever $\min\{\alpha(x, y)H(Tx, Ty), M(x, y)\} > 0$, where

$$\begin{aligned} M(x,y) &= \max\Big\{d(x,y), d(x,Tx) - d(A,B), d(y,Ty) - d(A,B), \\ &\quad \frac{d(x,Ty) + d(y,Tx) - 2d(A,B)}{2}\Big\} + L[d(y,Tx) - d(A,B)] \end{aligned}$$

and $L \geq 0$.

An existence result regarding a best proximity point of such a generalized contractive mapping was proved, under some additional requirements, with respect to strictly α-admissibility, the accomplishment of the weak P-property, continuity or some other kind of condition imposed on α.

Theorem 0.7 ([20], Theorem 2.5). *Let A and B be nonempty, closed subsets of X, and (X, d) a complete metric space. Assume that A_0 is nonempty and $T\colon A \to CL(B)$ is an F-α-proximal contraction of Ćirić type satisfying the following conditions:*

(i) *$Tx \subseteq B_0$ for each $x \in A_0$ and (A, B) satisfies the weak P-property;*

(ii) *T is strictly α-proximal admissible;*

(iii) *there exist $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$\alpha(x_0, x_1) > 1 \quad \text{and} \quad d(x_1, y_1) = d(A, B);$$

(iv) *T is continuous, or, for any sequence $\{x_n\} \subseteq A$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.*

Then T has a best proximity point.

In the next section of the chapter, by considering particular values for a_1, a_2, a_3 a_4 or L in Theorem 0.6, some well best proximity point results were retrieved. An example is provided to show the usability of Theorem 0.6. Some fixed point results have been obtained, as follows.

Theorem 0.8 ([20], Theorem 2.6). *Let (X, d) be a complete metric space. Assume $T\colon X \to CL(X)$ is a mapping for which there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$\tau + F(\alpha(x, y)H(Tx, Ty)) \leq F(N(x, y)),$$

for each x, $y \in X$, whenever $\min\{\alpha(x, y)H(Tx, Ty), N(x, y)\} > 0$, where

$$N(x, y) \;=\; a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + L d(y, Tx),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$. Further assume that the following conditions hold:

(i) *T is strictly α-admissible, that is, if $\alpha(x,y) > 1$, then $\alpha(a,b) > 1$ for each $a \in Tx$ and $b \in Ty$;*

(ii) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) > 1$;*

(iii) *T is continuous, or, for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.*

Then T has a fixed point.

Theorem 0.9 ([20], Theorem 2.7)**.** *Let (X,d) be a complete metric space. Assume $T\colon X \to CL(X)$ is a mapping for which there exist continuous F in $\mathfrak{F}$ and $\tau > 0$ such that*

$$\tau + F(\alpha(x,y)H(Tx,Ty)) \leq F(M(x,y)),$$

for each x, $y \in X$, whenever $\min\{\alpha(x,y)H(Tx,Ty), M(x,y)\} > 0$, where

$$M(x,y) \quad = \quad \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\right\} + Ld(y,Tx)$$

and $L \geq 0$. Further assume that the following conditions hold:

(i) *T is strictly α-admissible, that is, if $\alpha(x,y) > 1$, then $\alpha(a,b) > 1$ for each $a \in Tx$ and $b \in Ty$;*

(ii) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) > 1$;*

(iii) *T is continuous, or, for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.*

Then T has a fixed point.

The outcome of this part comprises two definitions, seven theorems, and nine corollaries.

In the third chapter, "Proximal contractions in gauge spaces", we have chosen the gauge spaces as framework for the development of best proximity results for adequate proximal contractions. The foundations of gauge spaces are pseudo metric spaces, namely if X is a nonempty set, a function $d\colon X \times X \to [0, \infty)$ is called a pseudo metric on X if for each x, y, $z \in X$, the following axioms hold:
(i) $d(x,x) = 0$;
(ii) $d(x,y) = d(y,x)$;
(iii) $d(x,z) \leq d(x,y) + d(y,z)$.

If $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$ is a family of pseudo metrics on X, the topology $\mathfrak{T}(\mathfrak{F})$ having as subbasis the family of balls

$$\mathfrak{B}(\mathfrak{F}) = \{B(x, d_v, \epsilon) : x \in X, d_v \in \mathfrak{F}, \text{ and } \epsilon > 0\}$$

is the topology induced by the family $\mathfrak{F}$ of pseudo metrics. The pair $(X, \mathfrak{T}(\mathfrak{F}))$ is called a gauge space. If we consider $\mathfrak{F}$ as being separating, note that $(X, \mathfrak{T}(\mathfrak{F}))$ is Hausdorff.

The first generalized contraction we introduced in this chapter was inspired by the notion of R-functions defined by Hierro and Shahzad [19]. For a nonempty set $D \subseteq \mathbb{R}$, a mapping $\varrho\colon D \times D \to \mathbb{R}$ is called an R-function if it satisfies the following two conditions:

(ϱ_1) If $\{a_n\} \subset (0,\infty) \cap D$ is a sequence such that $\varrho(a_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.

(ϱ_2) If $\{a_n\}, \{b_n\} \subset (0,\infty) \cap D$ are two sequences converging to the same limit $L \geq 0$ and verifying the inequality $L < a_n$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $L = 0$.

Sometimes we need an additional condition to the ones previously mentioned, more precisely

(ϱ_3) If $\{a_n\}, \{b_n\} \subset (0,\infty) \cap D$ are two sequences such that $b_n \to 0$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.

In the sequel, denote by R_D the family of R-functions whose domain is $D \times D$.

Based on this concept of R-functions, we defined the notion of R-proximal contraction of the first kind in the setting of gauge spaces.

Definition 0.6 ([20], Definition 3.11)**.** Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a complete gauge space induced by a family of pseudo metrics $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$. A mapping $T\colon A \to B$ is said to be an R-proximal contraction of the first kind if there exists an R-function $\varrho \in R_D$ such that $ran(\mathfrak{F}) \subseteq D$, where the range of $\mathfrak{F}$ is defined as $ran(\mathfrak{F}) = \{d_v(x,y) : x, y \in X \text{ and } v \in \mathfrak{A}\}$, and $d_v(u_1, Tx_1) = d_v(A,B) = d_v(u_2, Tx_2)$ imply that

$$\varrho(d_v(u_1,u_2), d_v(x_1,x_2)) > 0, \text{ for each } v \in \mathfrak{A},$$

where $u_1, u_2, x_1, x_2 \in A$, provided that $d_v(u_1,u_2)$, $d_v(x_1,x_2)$ are not null.

Another concept we introduced is that of $\mathcal{Z}$-proximal contractions of the first kind, as follows.

Definition 0.7 ([20], Definition 3.12)**.** A mapping $T\colon A \to B$ is said to be a $\mathcal{Z}$-proximal contraction of the first kind if there exists a simulation function function $\zeta\colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ such that $d_v(u_1, Tx_1) = d_v(A,B) = d_v(u_2, Tx_2)$ imply that

$$\zeta(d_v(u_1,u_2), d_v(x_1,x_2)) > 0, \text{ for each } v \in \mathfrak{A},$$

where u_1, u_2, x_1, $x_2 \in A$, provided that $d_v(u_1,u_2)$, $d_v(x_1,x_2)$ are not null.

An existence and uniqueness result with respect to best proximity points of R-proximal contractions of the first kind was stated, in some suitable conditions regarding the continuity of the involved operator or an approximately compactness condition.

Theorem 0.10 ([20], Theorem 3.1)**.** *Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a complete gauge space induced by a separating family of pseudo metrics $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$. Let A and B be nonempty, closed subsets of X, and A_0 be nonempty. Let $T\colon A \to B$ be an R-proximal contraction of the first kind with respect to $\varrho \in R_D$, such that $T(A_0) \subseteq B_0$. Further, assume that at least one of the following conditions hold:*

(a) *T is continuous;*

(b) *The function ϱ satisfies condition (ϱ_3), and B is approximately compact with respect to A.*

Then there exists a unique element $x \in A$ such that $d_v(x, Tx) = d_v(A,B)$, for each $v \in \mathfrak{A}$.

This theorem has some important consequences, such as existence and uniqueness results for the class of $\mathcal{Z}$-proximal contractions of the first kind, or the Geraghty proximal contractions of the first kind.

By slightly changing some of the components of Definition 0.6, and of Definition 0.7, we introduced the concept of R-proximal contraction of the second kind and of $\mathcal{Z}$-proximal contraction of the second kind, as follows.

Definition 0.8 ([20], Definition 3.13)**.** A mapping $T\colon A \to B$ is said to be an R-proximal contraction of the second kind if there exists an R-function $\varrho \in R_D$ such that $ran(\mathfrak{F}) \subseteq D$ and $d_v(u_1, Tx_1) = d_v(A, B) = d_v(u_2, Tx_2)$ imply that

$$\varrho(d_v(Tu_1, Tu_2), d_v(Tx_1, Tx_2)) > 0, \text{ for each } v \in \mathfrak{A},$$

where u_1, u_2, x_1 and x_2 belong to A, provided that $d_v(Tu_1, Tu_2)$, $d_v(Tx_1, Tx_2)$ are not null.

Definition 0.9 ([20], Definition 3.14)**.** A mapping $T\colon A \to B$ is said to be a $\mathcal{Z}$-proximal contraction of the second kind if there exists a simulation function function $\zeta\colon [0, \infty) \times [0, \infty) \to \mathbb{R}$ such that $d_v(u_1, Tx_1) = d_v(A, B) = d_v(u_2, Tx_2)$ imply that

$$\zeta(d_v(Tu_1, Tu_2), d_v(Tx_1, Tx_2)) > 0 \text{ for each } v \in \mathfrak{A},$$

where $u_1, u_2, x_1, x_2 \in A$, provided that $d_v(Tu_1, Tu_2)$, $d_v(Tx_1, Tx_2)$ are not null.

A best proximity property was proved with respect to R-proximal contractions of the second kind, provided a continuity assumption is also fulfilled.

Theorem 0.11 ([20], Theorem 3.2)**.** *Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a complete gauge space induced by a separating family of pseudo metrics $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$. Let A and B be nonempty, closed subsets of X such that A is approximately compact with respect to B and A_0 is nonempty. Let $T\colon A \to B$ be an R-proximal contraction of the second kind with respect to $\varrho \in R_D$. Further, assume that T is continuous and $T(A_0) \subseteq B_0$. Then there exists a unique element $x \in A$ such that $d_v(x, Tx) = d_v(A, B)$, for each $v \in \mathfrak{A}$.*

Corollaries regarding $\mathcal{Z}$-proximal contractions of the second kind, and also Geraghty proximal contractions of the second kind prove the usability and generality of the previous statement.

The second part of this chapter is based on implicit proximal contractions. In order to define them we used a (c)-comparison function and some conditions from the properties of functions κ in Chapter 1. More precisely, we denoted by Φ_ψ the family of functions $\varphi\colon (\mathbb{R}^+)^4 \to \mathbb{R}^+$ which satisfy the following conditions:

(i) φ is continuous and nondecreasing as a function of its first variable only and of its fourth variable only;

(ii) if p, $q \in \mathbb{R}^+$ such that if $p < q$ and $p \leq \varphi(q, q, p, q)$, then $p \leq \psi(q)$; if $p \geq q$ and $p \leq \varphi(p, q, p, p)$, then $p = 0$.

(iii) If $l \in \mathbb{R}^+$ such that $l \leq \varphi\left(0, 0, l, \frac{1}{2}l\right)$, then $l = 0$.

Based on this family, the implicit generalized contraction of the first kind was introduced.

Definition 0.10 ([20], Definition 3.15)**.** Let A and B be nonempty subsets of X, and $(X, \mathfrak{T}(\mathfrak{F}))$ a gauge space. A mapping $T\colon A \to B$ is called implicit generalized proximal contraction of the first kind if there exists $\varphi \in \Phi_\psi$ such that $d_v(u_1, Tx_1) = d_v(A, B) = d_v(u_2, Tx_2)$ imply

$$d_v(u_1, u_2) \leq \varphi\Big(d_v(x_1, x_2), d_v(x_1, u_1), d_v(x_2, u_2), \frac{1}{2}(d_v(x_2, u_1) + d_v(x_1, u_2))\Big),$$

for each $v \in \mathfrak{A}$.

By adding an approximately compactness condition, we formulated a best proximity point result, and proved it, by using fixed point methodologies.

Theorem 0.12 ([20], Theorem 3.3)**.** *Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a complete gauge space induced by a separating family of pseudo metrics $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$. Let A and B be nonempty, closed subsets of X such that B is approximately compact with respect to A and A_0 is nonempty. Let $T\colon A \to B$ be implicit generalized proximal contraction of the first kind with $T(A_0) \subseteq B_0$. Then T has a best proximity point, that is, there exists an element x in A such that $d_v(x, Tx) = d_v(A, B)$, for all $v \in \mathfrak{A}$.*

An example is given with regard to this theorem, then some classical results in best proximity theory are derived from previous theorems.

By changing the inequality which defined the implicit generalized proximal contraction of the first kind, the concept of implicit generalized contraction of the second kind aroused.

Definition 0.11 ([20], Definition 3.16)**.** Let A and B be nonempty subsets of X. A mapping $T\colon A \to B$ is called implicit generalized proximal contraction of the second kind if there exists $\varphi \in \Phi_\psi$ such that, for each $x_1, x_2, u_1, u_2 \in A$, $d_v(u_1, Tx_1) = d_v(A, B) = d_v(u_2, Tx_2)$ imply

$$\begin{aligned} d_v(Tu_1, Tu_2) \quad \leq \quad & \varphi\big(d_v(Tx_1, Tx_2), d_v(Tx_1, Tu_1), d_v(Tx_2, Tu_2), \\ & \frac{1}{2}(d_v(Tx_2, Tu_1) + d_v(Tx_1, Tu_2))\big), \end{aligned}$$

for each $v \in \mathfrak{A}$.

This type of contractive conditions allowed us again to provide best proximity results.

Theorem 0.13 ([20], Theorem 3.4)**.** *Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a complete gauge space induced by a separating family of pseudo metrics $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$. Let A and B be nonempty, closed subsets of X such that A is approximately compact with respect to B and A_0 is nonempty. Let $T\colon A \to B$ be a continuous implicit generalized proximal contraction of second kind such that $T(A_0) \subseteq B_0$. Then T has a best proximity point, that is, there exists an element x in A such that $d_v(x, Tx) = d_v(A, B)$, for all $v \in \mathfrak{A}$.*

Best proximity theorems on the framework of classic metric spaces are derived at the end of this chapter, to prove the generality of our theorems. The outcome of the chapter consists of six definitions, six theorems, and eleven corollaries.

Chapter 4, "Reckoning best proximity points", is dedicated to numerical schemes regarding the determination of best proximity points associated with adequate mappings. The numerical algorithms proposed here used mainly techniques similar to those in Thakur *et al.* [29], and those in the CQ algorithm of Nakajo and Takahashi [27] (see also Takahashi [28]), in suitable conditions.

The chapter begins with a numerical approach which characterizes the fixed points of mappings which fulfill the condition (E) introduced by García-Falset *et al.* [17]. If $(E, \|\cdot\|)$ is a Banach space, C a nonempty subset of E and $\mu \geq 1$, a mapping $T\colon C \to E$ satisfies the condition (E_μ) if, for all $x,\ y \in C$, $\|x - Ty\| \leq \mu\|x - Tx\| + \|x - y\|$; a mapping T satisfies the condition (E) whenever it fulfills the condition (E_μ) for some $\mu \geq 1$. By adapting the methodologies in Thakur et al. [29], the next algorithm for reckoning fixed points of mappings endowed with the condition (E) is proposed:

$$\left.\begin{aligned} x_1 &\in C \\ x_{n+1} &= Ty_n \\ y_n &= T\left(\left(1-\alpha_n\right)x_n + \alpha_n z_n\right) \\ z_n &= \left(1-\beta_n\right)x_n + \beta_n T x_n \end{aligned}\right\} \tag{1}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. A technical lemma is stated and proved, regarding the limit of a sequence defined by means of $\{x_n\}$ and a fixed point of the mapping involved; necessary and sufficient conditions for the nonemptiness of the set of the fixed points of such an operator with suitable properties are provided.

Lemma 0.1 ([20], Lemma 4.2)**.** *Let C be a nonempty, closed, convex subset of a Banach space $(E, \|\cdot\|)$, and let $T\colon C \to C$ be a mapping satisfying the condition* (E) *such that $F(T) \neq \emptyset$. For arbitrary chosen $x_1 \in C$, let the sequence $\{x_n\}$ be generated by the iterative process* (1)*. Then $\lim\limits_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(T)$.*

Theorem 0.14 ([20], Theorem 4.1)**.** *Let C be a nonempty, closed, convex subset of a uniformly convex Banach space E and let $T\colon C \to C$ be a mapping satisfying condition* (E)*. Given a point $x_1 \in C$, let the sequence $\{x_n\}$, $n \geq 1$, be generated by the iterative process* (1) *with $\{\alpha_n\}$ and $\{\beta_n\}$ bounded away from* 0 *and* 1*. Then $F(T) \neq \emptyset$ if and only if the sequence $\{x_n\}$ is bounded and $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$ (i.e. $\{x_n\}$ is an approximate fixed point sequence).*

Next, we introduced the definition of mappings endowed with the (EP) property and proposed an iterative procedure for the best proximity point reckoning with respect to such operators.

Definition 0.12 ([20], Definition 4.4)**.** Let $(E, \|\cdot\|)$ be a uniformly convex Banach space and (A, B) be a pair of nonempty, closed and convex subsets of E, satisfying also the P-property, and such that $A_0 \neq \emptyset$. Denote by $P_{A_0}\colon E \to A_0$ the metric projection onto A_0. A mapping $T\colon A \to B$ is said to satisfy the condition (EP) if and only if

$$\|x - P_{A_0}Ty\| \leq \mu\|x - P_{A_0}Tx\| + \|x - y\|, \text{ for all } x, y \in A.$$

Under suitable hypotheses regarding the sets A, and B, an algorithm was designed for the computation of best proximity points of these kind of operators. More specifically, if (A, B) is a pair of of nonempty, closed and convex subsets of E, which fulfills the P-property, with $A_0 \neq \emptyset$, where $(E, \|\cdot\|)$ is a uniformly convex Banach space, the numerical scheme is described in the next lines:

$$\left.\begin{aligned} x_1 &\in A_0 \\ x_{n+1} &= P_{A_0} T y_n \\ y_n &= P_{A_0} T\left(\left(1-\alpha_n\right) x_n+\alpha_n z_n\right) \\ z_n &= \left(1-\beta_n\right) x_n+\beta_n P_{A_0} T x_n \end{aligned}\right\} \tag{2}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences bounded away from 0 and 1.

An extension of Theorem 0.14 is provided for the case of mappings endowed with the condition (EP), characterizing the existence of best proximity points of such operators.

Theorem 0.15 ([20], Theorem 4.2)**.** *Let (A, B) be a pair of nonempty subsets of E, and $(E, \|\cdot\|)$ be a uniformly convex Banach space, where the pair has the P-property, A and B are closed and convex, and $A_0 \neq \emptyset$. Suppose the mapping $T\colon A \rightarrow B$ satisfies the condition* (EP) *with $T\left(A_0\right) \subseteq B_0$ and let $\{x_n\}$ be the sequence generated by the iterative process* (2)*. Then, the mapping T has a best proximity point if and only if $\{x_n\}$ is bounded and $\lim\limits_{n \rightarrow \infty}\|x_n-T x_n\|=d(A, B)$.*

This outcome has an important consequence on the algorithm from (2); it allowed us to obtain the strong convergence of the sequence $\{x_n\}$ to a best proximity point of the operator involved in the numerical scheme, in case the (EP) condition is fulfilled.

Corollary 0.1 ([20], Corollary 4.2)**.** *Let (A, B), T and $\{x_n\}$ be as in Theorem* 0.15 *and suppose additionally that A is compact. If $F(P_{A_0} T) \neq \emptyset$, then the sequence $\{x_n\}$ generated by the iterative process* (2) *converges strongly to a best proximity point of T.*

The last part of the chapter is focused on an algorithm which is a hybrid between the iterative process (2) and the CQ algorithm introduced by Nakajo and Takahashi [27]. Here we worked on a real Hilbert space H, with the inner product $\langle\cdot, \cdot\rangle$, and the norm $\|\cdot\|$; A and B are nonempty, closed and convex subsets of H, endowed with the P-property, $T\colon A \rightarrow B$ is an operator whose set of best proximity points is A_T.

Our proposed algorithm,

$$\left.\begin{aligned} & x_1 \in A_0 \text{ arbitrary}, \\ & z_n=\left(1-\beta_n\right) x_n+\beta_n P_{A_0} T x_n, \\ & y_n=P_{A_0} T\left(\left(1-\alpha_n\right) x_n+\alpha_n z_n\right), \\ & w_n=P_{A_0} T y_n, \\ & Q_n=\left\{u \in A_0\colon \left\langle x_n-u, x_n-x_1\right\rangle \leq 0\right\}; \\ & C_n=\left\{u \in A_0\colon \max \left\{\|w_n-u\|,\|y_n-u\|,\|z_n-u\|\right\} \leq\|x_n-u\|\right\}, \\ & x_{n+1}=P_{\left(C_n \cap Q_n\right)} x_1, \end{aligned}\right\} \tag{3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences bounded away from 0 and 1, is strongly convergent to a best proximity point of T, when this operator satisfies the condition (EP) and some suitable hypotheses are accomplished.

Theorem 0.16 ([20], Theorem 4.3)**.** *Let (A, B) a pair of nonempty, closed and convex subsets of a real Hilbert space and suppose the pair has the P-property. Let $T\colon A \to B$ be a mapping which satisfies the condition* (EP) *such that A_T is a nonempty, closed, convex subset of A_0 and $T(A_0) \subset B_0$. Then, the sequence $\{x_n\}$, generated by the algorithm from* (3)*, converges to a best proximity point. In particular, it converges to p, where $p = P_{A_T}(x_1)$. Moreover, the same holds true for the sequences $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$.*

Our outcome in this chapter comprises one definition, two algorithms, three theorems, and two corollaries.

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