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Some Classes of Nonlinear Operators for Fixed Point Problems with Applications

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Abstract of the PhD Thesis

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Contents

Abstract

General objective. This study is motivated by the necessity of finding general classes of operators suitable for fixed point problems, along with versatile iteration procedures to make possible the numerical reckoning of solutions, once the existence is postulated. This thesis aims to provide a consistent approach of the three–step iteration procedure S_n. The general direction of study is nonlinear analysis, and the main results are related to fixed point theory with applications based on numerical modeling. The main conclusions resulting from the study of this iterative process are related to several features:

- *qualitative aspects (data dependence and stability).* Iterative procedures are used in practice to provide algorithms for determining solutions to all sorts of nonlinear problems. The running of the algorithms is subject to perturbations induced by the limitations of the computer system. Therefore, we must constantly ensure that the approximations made during the running of the algorithm do not dramatically affect the estimation of the solution. A qualitative analysis of an iterative process in general, and of the S_n process in particular, is motivated by such practical reasons.

- *instrumental value.* One main concern regarding the iteration procedure subject to analysis is related to the convergence of the sequence of iterations towards the solution of the studied problem. Most of the time, the results obtained refer to weak convergence. However, by introducing additional restrictions for the underlying metric setting or for the operator involved in the iterative process, strong convergence results are also obtained.

- *versatility.* The iterative procedure S_n is adapted to various working environments: Hilbert spaces, uniformly convex Banach spaces, modular structures or curved geometric spaces. Also, different classes of generalized nonexpansive operators are used: operators with property (D), Garcia-Falset operators (satisfying condition (E)), firmly-nonexpansive operators, metric projections, or nonspreading mappings.

Methodology. The methods used to perform the analysis of the S_n iteration procedure are varied: the study of convergence relies on the uniqueness of the asymptotic center. Stability is analyzed according to the pattern provided by Harder and Hiks [14], and data dependence analysis uses the method initiated by Rus and Mureşan [29]. The comparative analysis regarding the efficiency of the studied process in relation to other procedures is performed using polynomiographic techniques. Polar coordinates are used to find the proper expression of the the iterative process on the Poincaré half-plane. The algorithms used in numerical applications are run using Matlab.

General state of the art. It is a well-known fact that various problems of applied mathematics could be generally expressed by means of systems of equations. Obviously, reaching the exact solutions is the main goal. However, quite often, the mathematical tools are not enough developed to provide the proper answer for this issue. For this reason, sometimes, we are content with less: we aim to confirm at least the existence and, possibly, the uniqueness of the solution. Finding an approximation of the solution would also be an important achievement, especially if this determination could be done numerically. A powerful instrument in this regard is the Contraction Principle of Banach. It requires, first of all, expression of the problem as a fixed point equation, in a properly defined setting (usually a complete metric space) and, secondly, evaluating the iteration function T as being a contraction. In addition to the existence and uniqueness statement, Banach's Contraction Principle also points out that the fixed points of a contraction can be obtained using Picard iterations.

However, the contractive property is sometimes too restrictive. This is the reason why, in the past 50 years, the study of fixed points for more general classes of mappings has become an important and very active research direction. However, unlike contractions, successive iterations for nonexpansive mapping do not necessarily converge at a fixed point. So there is a major limitation of the Picard iteration sequence under the aspect of reaching the fixed point. That is why the study of (generalized) nonexpansiveness conditions came with the necessity of providing new suitable iteration procedures. One pioneer result for the approximation of fixed points for a nonexpansive mapping was established by Krasnosel'skii [21]. The result shows that if X is a uniformly convex Banach space and $T\colon X \to X$ is a nonexpansive mapping, then the successive iterations of the function $\frac{1}{2}(I+T)$ are convergent to a fixed point of T. Other important iterative schemes were defined by Mann [23] (formally this is a generalization of the Krasnosel'skii iteration resulted by replacing the iteration parameter with a sequence of real numbers), Ishikawa [15] (to reckon the fixed points of Lipschitzian pseudocontractive mappings on

Hilbert spaces), Noor (a valuable three-step iteration procedure intended primarily to solve variational inequalities), Sahu *et al.* (a new iteration technique for solving convex programming and split feasibility problems), and the list may go on.

An interesting aspect is the way these iteration procedures interfere with all sorts of nonlinear operators. For instance, Suzuki [32] proved a convergence result for a mapping satisfying condition C using a Krasnosel'skii iterative process. Thakur *et al.*. [34] used a three-step iteration process for approximating a fixed point of nonexpansive mappings. In 2016, Thakur *et al.* [33] introduced a three-step iteration process by means of two parametric sequences, and connected this procedure to Suzuki-type mappings. Soon after, Sintunavarat and Pitea [31] introduced the S_n iterative scheme, in connection with Berinde-type operators. The latter is precisely the iteration procedure undergoing in-depth research during this thesis.

All in all, the definition and use of iterative procedures has been synchronized with the most recent generalizations of nonexpansivity. A first major generalized nonexpansiveness condition, which later set the tone for extensive studies in this direction, was defined by Suzuki [32], which introduced the so-called condition (C) on Banach spaces. Operators which fulfill this property, hereinafter referred to as Suzuki mappings, are more general than the nonexpansive operators, but remain subordinate to the class of quasinonexpansive mappings. Subsequently, Garcia-Falset *et al.* [13] proposed two new possible extensions for condition (C); the first one was called condition (E), and was inspired by a certain property of Suzuki-mappings; the second generalized nonexpansivity was a direct extension of Suzuki's definition, resulting the so-called condition (C_λ). Other important generalized nonexpansivity property inspired by Suzuki's condition were provided by Pant *et al.* [27]. The resulting α-Reich-Suzuki nonexpansive mappings are contained in the class of mappings satisfying condition (E). Bejenaru and Postolache [3] also contributed with a new nonexpansivity condition, condition (D) proving further that the operators with property (D) on Banach spaces satisfy also condition (E).

Finally, existing studies use a variety of metric structures. The study of iterative processes related to the determination of fixed points for certain classes of nonexpansive operators are carried out both on Banach spaces, enriched with uniformly convex structure (see [13],[22],[24],[27]), and on more general metric spaces, such as convex modular structures (see [3],[17]) or on curved metric spaces (see [11],[28]). In terms of feasibility issues, variational inequalities, variational inclusions, the study of maximum monotone operators and so on, the usual framework is that of Hilbert spaces or Banach spaces with enriched topological properties (see [4],[5]-[7],[12],[19],[30]).

Thesis description: structure and content.

In Chapter 1, **Operators with condition (D) in uniformly convex spaces** [8, 10], we analyze the recently introduced nonexpansivity condition, the so-called condition (D). It was introduced initially by Bejenaru and Postolache [3] and the formal definition requires that a selfmapping T on a nonempty subset of C a Banach space to satisfy the inequality

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x \in C$ and $y \in C(T, x)$, where

$$C(T, x) = \{Tp : p \in C, \|Tp - p\| \leq \|Tx - x\|\}.$$

In Lemma 1 and Lemma 2 we state and proof basic properties of operators with condition (D). Example 1 and Example 2 analyze the relationship between mappings satisfying conditions (C) and (D), respectively. These prove that the mapping class that meets the (D) condition is not included in the Suzuki mapping class with condition (C) and so emphasizes the relevance of conducting a study of operators with property (D).

Lemma 1. *If T satisfies condition* (D)*, then:*

$$||x - Ty|| \leq 3||Tx - x|| + ||x - y||, \ \forall x, y \in C.$$

Lemma 2. *Let T be a mapping on a subset C of a Banach space X with the Opial property. Assume that T satisfies condition* (D)*. If the sequence $\{x_n\}$ converges weakly to z and $\lim\limits_{n\to\infty} \|Tx_n - x_n\| = 0$, then $Tz = z$.*

Example 1. Consider the mapping

$$T\colon [0,1] \to [0,1], \quad Tx = \begin{cases} 1, & x = 0 \\ \frac{1}{2}, & x \in \left(0, \frac{1}{2}\right) \\ x, & x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

A carefully conducted survey proves that T defined above is not a Suzuki mapping, but satisfies condition (D).

Example 2. Let us consider the Banach space $X = L^\infty(\mathbb{R})$ with the essential supremum norm $\|f\|_\infty = \text{ess sup}_{\mathbb{R}} |f| = \inf\{M : |f(x))| \leq M$ almost everywhere on $\mathbb{R}\}$. Let C be the set of all the functions $f\colon \mathbb{R} \to [0, 11]$, satisfying $f(x) = f(0)$, for all $x \leq 0$, and define the mapping

$$T\colon C \to C, \quad Tf(x) = \begin{cases} f(x), \quad x > 0 \\ \frac{4}{11} f(0), \quad x \leq 0, \quad f(0) \neq 11 \\ 5, \quad x \leq 0, \quad f(0) = 11. \end{cases}$$

Then T is not a nonexpansive mapping, but satisfies both conditions (C) and (D). It is clear that the classes of mappings meeting conditions (C) and (D), respectively are not completely disjoint, since they both include the class of nonexpansive operators. However, this example proves that there are also common elements that are not necessarily nonexpansive.

Further on, this chapter also includes a convergence survey conducted based on the iteration procedure S_n, for operators with property (D). This iterative scheme was introduced in 2016 by Sintunavarat and Pitea [31] in connection with Berinde-type operators. For an arbitrary $x_1 \in C$, the sequence $\{x_n\}$ results from the three-step procedure

$$\begin{cases} x_{n+1} = (1-\alpha_n)\,Tz_n + \alpha_n T y_n \\ z_n = (1-\gamma_n)\,x_n + \gamma_n y_n \\ y_n = (1-\beta_n)\,x_n + \beta_n T x_n, \end{cases} \tag{1}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences $(0,1)$.

Overall, the convergence analysis performed here adopts a similar type of reasoning as in Thakur *et al.* [33]. Lemma 3, Theorem 1, Theorem 2 and Theorem 3 of the present chapter phrase and prove the main convergence outcomes.

Lemma 3. *Let C be a nonempty convex subset of a Banach space X and let $T\colon C \to C$ be a mapping satisfying condition* (D) *with $F(T) \neq \emptyset$. For an arbitrary $x_1 \in C$, let the sequence $\{x_n\}$ be generated by* (1)*. Then, $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(T)$.*

Theorem 1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $T\colon C \to C$ be a mapping satisfying condition* (D)*. For an arbitrarily chosen $x_1 \in C$, let the sequence $\{x_n\}$ be generated by* (1) *for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of real numbers in $[a,b]$ for some a and b with $0 < a \leq b < 1$. Then, $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Theorem 2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X with the Opial property, and let T and $\{x_n\}$ be as in Theorem* 1*, with the additional assumption $F(T) \neq \emptyset$. Then, $\{x_n\}$ converges weakly to a fixed point of T.*

Theorem 3. *Let C be a nonempty, compact, and convex subset of a uniformly convex Banach space X, and let T and $\{x_n\}$ be as in Theorem* 1*. If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to a fixed point of T.*

The chapter is completed with a qualitative analysis of the iterates under S_n such as stability and data dependence, which follows closely the formal modeling proposed by of

Harder and Hicks [14] or Rus and Mureşan [29]. Broadly speaking, an iteration process that converges to a unique fixed point is stable if the convergence of the procedure is not influenced by the numerical errors that occur during each iteration step. On the other side, the motivation for data dependence analysis is given by the fact that the practical implementation of algorithms works with approximations instead of theoretical, unperturbed operators. The data dependence analysis aims to answer the following question: To what extent is the achievement of the real fixed point affected by the use of a disturbed operator? In other words, by deviating from the actual mapping T to a perturbed mapping $\tilde{T}$, the numerical simulation should keep the output close enough to the actual solution. Obviously, the errors would reach a minimum level when the procedure would depend only on the initial estimate and not on the operator itself.

The main results of our survey are included in Theorem 4, Theorem 5 and Theorem 6. Theorem 4 is meant to prove that, for contractive mappings, the iteration procedure S_n (1) really converges to the unique fixed point of T, while Theorem 5 states and proves its stability. Finally, Theorem 6 provides an estimate of the deviation of the fixed point in terms of the maximum admissible error, also proving the data independence of the iteration procedure S_n.

Theorem 4. *Let C be a nonempty closed convex subset of a Banach space X and let $T\colon C \to C$ be a contraction mapping. Let $\{x_n\}$ be an iterative sequence generated by* (1)*, with $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$, satisfying $\sum_{n=1}^{\infty} \alpha_n\beta_n\gamma_n = \infty$. Then, $\{x_n\}$ converges strongly to the unique fixed point of T.*

Theorem 5. *Let C be a nonempty closed convex subset of a Banach space X and let $T\colon C \to C$ be a contraction mapping. Let $\{x_n\}$ be an iterative sequence generated by* (1)*, with $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ in $(0,1)$ such that $\alpha_n\beta_n\gamma_n \geq a > 0$, $\forall n$. Then, the iterative procedure* (1) *is T-stable.*

Theorem 6. *Let C be a nonempty closed convex subset of a Banach space X and let $T\colon C \to C$ be a contraction mapping with fixed point p. Let $\tilde{T}$ be an approximate mapping of the contraction mapping T with maximum admissible error ε, let $\{x_n\}$ be an iterative sequence generated by* (1)*, and define an iterative sequence $\{\tilde{x}_n\}$ as follows*

$$\begin{cases} \tilde{y}_n = (1-\beta_n)\,\tilde{x}_n + \beta_n\tilde{T}\tilde{x}_n \\ \tilde{z}_n = (1-\gamma_n)\,\tilde{x}_n + \gamma_n\tilde{y}_n \\ \tilde{x}_{n+1} = (1-\alpha_n)\,\tilde{T}\tilde{z}_n + \alpha_n\tilde{T}\tilde{y}_n, \end{cases}$$

for an arbitrary $\tilde{x}_1 \in C$, with real number sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ in $(0,1)$, satisfying $s_n = \beta_n(\alpha_n + \gamma_n - \alpha_n\gamma_n) \geq \dfrac{1}{\lambda - \theta}$ for some $\lambda > \theta$.

If $\lim_{n\to\infty} \tilde{x}_n = \tilde{p}$*, then*

$$\|p - \tilde{p}\| \leq \frac{\lambda\varepsilon}{1-\theta}.$$

Let us note the fact that Theorem 6 provides two control parameters for the deviation from the solution: the maximum admissible error ε of the approximate operator $\tilde{T}$ and λ, which is a control element for the iteration coefficients. More precisely, we note that s_n and λ are inversely proportional, while the deviations $\|p - \tilde{p}\|$ and λ are directly proportional. By rewriting $s_n = \beta_n[1 - (1-\alpha_n)(1-\gamma_n)]$, we notice that we can improve the performance of the algorithm (i.e., the distance $\|p - \tilde{p}\|$ should be as small as possible) by taking α_n, β_n, and γ_n close enough to 1.

Chapter 2 titled **Partially projective algorithm for split feasibility problem** [2, 8] introduces a new three-step projective algorithm that aims to solve the split feasibility problem (SFP). This particular problem, which relates to many problems in nonlinear optimization, such as fixed point problems, variational inequality problems, equilibrium problems, etc. was first considered by Censor and Elfving [7] in 1994. The general setting for this problem relies on two real Hilbert spaces H_1 and H_2, two closed, convex, and nonempty subsets C and Q of H_1 and H_2, respectively, and a bounded and linear operator $A\colon H_1 \to H_2$. This problem can be mathematically described as the procedure of searching a point $x \in H_1$ such that $x \in C$ and $Ax \in Q$.

Censor and Elfving [7] introduced also some appropriate algorithms for solving a class of inverse problems. At that time, however, their work did not gain much attention due to a major disadvantage: their algorithm required matrix inversion at each step of the iteration. As an alternative to their numerical procedure, in [5] and [6], Byrne suggested a new iterative method meant to solve the SFP, called the CQ method, which uses the orthogonal projections onto C and Q (subsets of the Euclidean arithmetic spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively). This represented a major improvement over Censor and Elfving achievement, since it does not involve the matrix inversion anymore, but only projections onto closed and convex subsets.

Algorithm 1 (CQ)**.** For an arbitrarily chosen initial point $x_0 \in H_1$, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = P_C\left[I - \gamma A^T(I - P_Q)A\right]x_n, \quad n \geq 0,$$

where by P_C and P_Q we denoted the projections onto the sets C and Q, respectively, and $0 < \gamma < \frac{2}{\rho(A^TA)}$, with $A^T\colon H_2 \to H_1$ being the transpose of A and $\rho\left(A^TA\right)$ the spectral radius of the selfadjoint operator A^TA.

If we think to a more general context and consider that H_1 and H_2 are infinite dimensional Hilbert spaces, then A^T must be replaced by the adjoint operator A^*. In this case, however, the strong convergence of CQ algorithm does not usually hold.

One possible way to approach the split feasibility problem relies on rephrasing it as a fixed point problem. This opens up to several possibilities in determining solutions by using iterative procedures. For instance, in 2010 Wang and Xu [37] pointed out that the CQ algorithm could be regarded as a special case of the Krasnosel'skii-Mann algorithm for approximating fixed point of nonexpansive mappings. This form however ensures only the weak convergence of this iterative process towards a solution point of the SFP. Further on, many other iterative procedures extended the idea above. The authors have been mainly preoccupied in providing less and less restrictive conditions to overcome the disadvantage in [37] and ensure a strong convergence (see, for instance, [12], [36]).

Given these historical aspects and inspired by recent papers [4, 12, 36] providing algorithms for solving SFP based on the TTP iteration scheme ([33]), we introduce a new algorithm for the split feasibility problem, which relies on the iteration procedure S_n (1). We shall refer to this algorithm as partially projective S_n iteration procedure (PPS_n), since it runs as a classical S_n iterative scheme, except the last step, where a projection is included.

Algorithm 2 (PPS_n)**.** For an arbitrarily chosen initial point $x_0 \in C$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} x_{n+1} = P_C\left[(1-\alpha_n)Sz_n + \alpha_n S y_n\right] \\ z_n = (1-\gamma_n)x_n + \gamma_n y_n \\ y_n = (1-\beta_n)x_n + \beta_n S x_n, \end{cases}$$

where

$$S\colon H_1 \to H_1,\ S = I - \frac{2}{\|A\|^2}A^*(I-P_Q)A.$$

and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three real sequences in $(0,1)$.

Turning back to operator S which is involved in Algorithm 2, we state and prove several properties, included in Lema 4 and Lemma 5.

Lemma 4. *If Ω is the solution set of the feasibility problem, than $\Omega = F(P_C S) = F(S) \cap C$.*

Lemma 5. *The mapping S is nonexpansive.*

Further on a result concerning the weak convergence of Algorithm 2 towards a solution point of the SFP is provided by Theorem 7, whose proof relies on the instrumental Lemmas 6 and 7.

Lemma 6. *Let* $\{x_n\}$ *be the sequence generated by Algorithm* 2*. Then,* $\lim_{n\to\infty} \|x_n - p\|$ *exists for any* $p \in \Omega$*.*

Lemma 7. *Let* $\{x_n\}$ *be the sequence generated by Algorithm* 2 *with* α_n*,* $\beta_n \in [p, q] \subset (0, 1)$*, for all* $n \geq 1$*. Then,* $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$*.*

Theorem 7. *Let* $\{x_n\}$ *be the sequence generated by Algorithm* 2 *with* α_n*,* $\beta_n \in [p, q] \subset (0, 1)$*, for all* $n \geq 1$*. Then,* $\{x_n\}$ *is weakly convergent to a point* $p \in \Omega$*.*

Moreover, by means of the nonexpansive mapping $T = P_C \circ S$ and under additional assumptions regarding it, some strong convergence outcomes are provided in Theorem 8 and Theorem 9.

Theorem 8. *Let* $\{x_n\}$ *be the sequence generated by Algorithm* 2 *with* α_n*,* $\beta_n \in [p, q] \subset (0, 1)$*, for all* $n \geq 1$*. Then,* $\{x_n\}$ *is strongly convergent to a point in* Ω *if and only if* $\liminf_{n\to\infty} d(x_n, \Omega) = 0$*, where* $d(x_n, \Omega) = \inf_{p\in\Omega} \|x_n - p\|$*.*

Theorem 9. *Let* $\{x_n\}$ *be the sequence generated by Algorithm* 2 *with* α_n*,* $\beta_n \in [p, q] \subset (0, 1)$*, for all* $n \geq 1$*. If* $T = P_C \circ S$ *satisfies Condition (A), then* $\{x_n\}$ *is strongly convergent to a point in* Ω*.*

The last part of this chapter provides several examples with computer simulations. Example 3 in particular leads to a visual comparison between the CQ and the PPS_n procedures.

Example 3. We consider the split feasibility problem assimilated to the following data: $H_1 = \mathbb{R}^2$ and $H_2 = \mathbb{R}^3$; the closed and convex subsets C and Q are the l_1-unit disk in $\mathbb{R}^2$ and l_2-unit ball in $\mathbb{R}^3$, respectively; the bounded linear operator is defined by the 3×2 matrix

$$A = \begin{pmatrix} -10 & 0 \\ 4 & -12 \\ -1 & 11 \end{pmatrix};$$

The role of this simulation is to decide which of the CQ algorithm or the PPS_n algorithm requires less iterative steps. Moreover, to obtain a global perspective on the efficiency of the two algorithms, we not only select an arbitrary initial estimate x_0, but evaluate the entire set C. To do so, we borrow a technique originally used to numerically determine the roots of complex polynomials, called polynomiography (please see, [16]). The result is a visual image of the convergence behavior for the two algorithms, which allows us to compare their efficiency and to conclude that the PPS_n algorithm is faster convergent than the CQ. Moreover, the resulting pictures emphasize also the solution set of the feasibility problem.

Example 4 emphasizes the idea of finding acceptable fitting lines for a set of data, when a maximum acceptable deviation δ is assumed. The major contribution of this chapter is to reformulate this interpolation problem as a problem of split feasibility, thus providing numerical methods to solve it. It is worth pointing out that these acceptable fitting lines are not necessarily the so-called "best fitting lines" from regression theory. There is however coincidence, when the accepted deviation is small enough.

Example 4. Suppose the data points are included in the table:

x	1	2	3	4
y	1	3	5	8

We wish to find a fitting line $f(t) = a + bt$, such that the deviations of the predicted values $f(t_i)$ from the exact measurements y_i, $i = 1, \ldots, 4$ not to exceed a permissible deviation δ. This problem is rewritten first as a split feasibility issue, by taking the entries:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}^t; \; x = (a,b)^t; \; y = (1,3,5,8)^t; \; Q = y + [-\delta, \delta]^4.$$

In addition, the origin $x_0 = (0,0)$ is considered as initial estimate and PPS_n algorithm is applied to reach a solution. As result we find the fitting lines for various assign values of the acceptable deviation δ.

At the end of this chapter, we rephrase the finite difference method for a boundary value problem (BVP) of type:

$$y'' + a(x)y' + b(x)y = c(x), \quad x \in [a,b], \; y(a) = y_a, \; y(b) = y_b$$

as a split feasibility problem, under the assumption of controlled truncation errors. As an application of such an approach, Example 5 performs a numerical simulation on a particular BVP in order to emphasize the advantages of the newly introduced procedure over the classical CQ algorithm.

Example 5. Let us consider the boundary value problem

$$y'' + xy' + y = 3x^2 + 2, \quad x \in [0,1]; \; y(0) = 0; \; y(1) = 1.$$

First, remark that the function $y(x) = x^2$ provides an exact solution. To obtain the discrete form, we take $h = \dfrac{1}{4}$ and $x_i = \dfrac{i}{4}$, $i = 0, 1, \ldots, 4$ and applying the finite difference method with included errors, we reach the split feasibility problem

$$\begin{pmatrix} -496 & 264 & 0 \\ 60 & -124 & 68 \\ 0 & 232 & -496 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in Q,$$

where

$$Q = (35, 11, -221) + \varepsilon \cdot [-5/4, 5/4] \times [-6/4, 6/4] \times [-7/4, 7/4].$$

First, we compare the number of iteration steps to be performed and the running time of the partially projective S_n algorithm for two values assigned to the truncation error. As expected, the smaller the deviation, the more accurate the solution. Secondly, we run both PPS_n and CQ Algorithms in order to compare them. Once more, the PPS_n procedure is proven to be significantly more advantageous than the CQ algorithm: the number of iteration steps is highly diminished and the estimated solution is closer to the exact one.

In Chapter 3 **Nonspreading mappings in modular vector spaces** [8, 9] we introduce the notion of a nonspreading mapping in the setting of modular vector spaces and suggest a way to reckon their fixed points.

The beginning of modular analysis was given by some practical examples of generalized function and sequence spaces provided by Orlicz and Birnbaum in the early 1930's. A deep analysis regarding modular function spaces and their suitability for fixed point theory was realized by Kozlowski (1988) in [20] and by Khamsi and Kozlowski in [18]. Still, the formal definition of modular vector spaces (not necessarily function-type spaces), as it is known and used today, was settled by Orlicz and Musielack in [25] and [26]. From that moment on, the modular setting became an interesting and nontrivial alternative to classical Banach spaces. Recent papers, using this particular framework as underlying setting are related with various modular nonexpansiveness conditions, please see Kassab and Ţurcanu [17].

In 2008, Kohsaka and Takahashi [19] introduced a new class of operators on Banach spaces, namely the nonspreading mappings. This way, they generalized the class of firmly nonexpansive type mappings. An interesting fact about the newly introduced operators concerns their appearance on Hilbert spaces. Starting from this particular expression, we adapt the definition of Kohsaka and Takahashi to convex modular vector spaces, and provide some properties of this class of operators. More specific, in Definition 1 we introduce the notion of modular nonspreading mapping with illustration in Example 6. Throughout the entire chapter we assume that ρ is a convex modular satisfying the Δ_2-condition and we use μ to denote the corresponding the modular factor.

Definition 1. Let C be a nonempty subset of a modular space X_ρ. A mapping $T\colon C \to X_\rho$ with

$$(1+\mu^2)\mu^2\rho^2(Tx-Ty) \le \rho^2(Tx-y)+\rho^2(x-Ty),$$

for all $x, y \in X_\rho$ is called a *modular nonspreading mapping.*

Example 6. In $\mathbb{R}$, we consider the function

$$\rho\colon \mathbb{R} \to [0,\infty),\ \ \rho(x) = |x|\sqrt{|x|},$$

which clearly defines a convex modular with modular factor $\mu = \sqrt{2}$.

Then, $T\colon \mathbb{R} \to \mathbb{R}, \quad Tx = \frac{x}{2}$ is proven to be a modular nonspreading mapping.

Further on, Lemma 8, Lemma 9 and Proposition 1 include some characteristic properties of the newly introduced class of operators.

Lemma 8. *Let C be a nonempty subset of a modular space X_ρ and let $T\colon C \to C$ be a modular nonspreading mapping with $F(T) \neq \emptyset$. Then T is a modular quasi-nonexpansive mapping (i.e. $\rho(Tx-p) \le \rho(x-p),\ \forall x \in C,\ p \in F(T)$).*

Lemma 9. *Let C be a nonempty ρ-bounded subset of a modular space X_ρ and $T\colon C \to C$ a modular nonspreading mapping. If $\{x_n\}$ is a sequence in C such that $\lim\limits_{n\to\infty}\rho(Tx_n - x_n) = 0$, and τ is the ρ-type function of $\{x_n\}$, then:*

(i) *$\tau(Tx) \le \tau(x)$, for all $x \in X_\rho$;*

(ii) *T leaves the minimizing sequences invariant (i.e. if $\{c_n\}$ is a minimizing sequence for τ, then so is $\{Tc_n\}$).*

Proposition 1. *Let C be a nonempty, convex and ρ-closed subset of a ρ-complete modular space X_ρ. Assume that ρ is* (UUC1) *and satisfies Fatou property. Consider the ρ-type function $\tau\colon C \to [0,\infty]$ of a sequence $\{x_n\} \subset X_\rho$ and suppose $\tau_0 = \inf\limits_{x\in C}\tau(x) < \infty$. Let $\{c_n\}$ and $\{d_n\}$ be two minimizing sequence for τ. Then,*

(i) *any convex combination of $\{c_n\}$ and $\{d_n\}$ is a minimizing sequence for τ as well;*

(ii) $\lim\limits_{n\to\infty}\rho(c_n - d_n) = 0$.

Last but not least, we establish fixed point results of modular nonspreading mappings in Lemma 10, Theorem 10 and Theorem 11. More precisely, we evaluate the solutions of fixed point equations involving this kind of operators based on the S_n (1) iterative process. The aim is to prove that the resulting iteration sequence, regardless of the initial estimate, is an approximating fixed point sequences and ultimately to provide conditions for convergence toward a fixed point of the nonspreading mapping.

Lemma 10. *Let C be a nonempty ρ-bounded and convex subset of X_ρ and let $T\colon C \to C$ be a modular nonspreading mapping with $F(T) \neq \emptyset$. For an arbitrary chosen $x_1 \in C$, let the sequence $\{x_n\}$ be generated by the iterative process* (1).

Then, $\lim_{n\to\infty} \rho(x_n - p)$ *exists for any* $p \in F(T)$.

Theorem 10. *Let X_ρ be a ρ-complete modular space and C be a nonempty convex ρ-closed and ρ-bounded subset of X_ρ. Suppose ρ is* (UUC1) *and satisfies Fatou property. Let $T\colon C \to C$ be a modular nonspreading mapping and let the sequence $\{x_n\}$ be generated by the iterative process* (1) *with $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ bounded away from* 0 *and* 1. *Then, $F(T) \neq \emptyset$ if and only if* $\lim_{n\to\infty} \rho(x_n - Tx_n) = 0$.

Theorem 11. *Let C be a nonempty ρ-compact and convex subset of a complete modular space X_ρ and let ρ, T and $\{x_n\}$ be as in Theorem* 10. *Then, the sequence $\{x_n\}$ ρ-converges to a fixed point of T.*

In Chapter 4, **Operators with property (E) in CAT(0) spaces** [1, 8], we provide a convergence analysis for the S_n iteration procedure in connection with two Garcia-Falset mappings, properly adapted for the non-positively curved setting, looking to obtain approximate common fixed points.

A CAT(0) space is a specific type of geodesic metric space, where every geodesic triangle in is at least as 'thin' as its comparison triangle in the Euclidean plane. This property is formally encrypted in the so-called CAT(0) inequality and ultimately stands as source for the non-positive curvature. Henceforth, the geometry of CAT(0) spaces exceeds any linear framework and goes toward a curved setting. We note in particular that all the pre-Hilbertian spaces, as well as the hyperbolic spaces $\mathbb{H}^n$ or the $\mathbb{R}$-trees, provide nontrivial examples of CAT(0) spaces.

As mentioned previously, in [13], Garcia-Falset *et al.*. introduced two distinct, generally unrelated nonexpansivity conditions, both intended primarily to generalize Suzuki's condition (C): condition (E_μ) and condition (C_λ), respectively. Each of them generated important developments in fixed point theory and not only. Conditions of type (E_μ) in particular were analyzed in connection with other nonexpansivity properties [27], various iterative processes including polynomiographic techniques [35], or in connection with signal recovery problems [24]. In a CAT(0) space (X, d), this property could be naturally defined as follows: a mapping $T\colon C \to X$ satisfies condition (E_μ) provided that, for all $x, y \in C$, the following inequality holds true: $d(x, Ty) \leq \mu d(x, Tx) + d(x, y)$.

An important property of CAT(0) spaces states that for each two given points x, y and each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = td(x, y)$ and

$d(y,z) = (1-t)d(x,y)$ (see [11]). This unique point is usually denoted by $(1-t)x \oplus ty$. With the help of it and by means of two operators T and S satisfying condition (E), we adapt the procedure S_n to a CAT(0) setting as follows: for an arbitrary $x_1 \in C$, let $\{x_n\}$ be generated by the procedure below

$$\begin{cases} x_{n+1} = (1-\alpha_n)\, Sz_n \oplus \alpha_n Ty_n \\ z_n = (1-\gamma_n)\, x_n \oplus \gamma_n y_n \\ y_n = (1-\beta_n)\, x_n \oplus \beta_n Sx_n, \end{cases} \tag{2}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$.

The results contained in Lemma 11 and Lemma 12 provide basic results of iterates of (E)-operators under S_n, while in Theorem 12 and Corollary 1, Δ-convergence results are stated and proven. Moreover, under additional assumptions regarding the involved mappings, some strong convergence results are obtained in Theorem 13 and Corollary 2.

Lemma 11. *Let C be a nonempty convex subset of complete CAT(0) space X. Let $T\colon C \to C$ and $S\colon C \to C$ be mappings satisfying condition (E) with $F \neq \emptyset$. Let $\{x_n\}$ be an iteration process described by* (2). *Then, $\lim_{n\to\infty} d(x_n, p)$ exists, for all $p \in F$, where F denotes the set of common fixed points for the operators S and T.*

Lemma 12. *Let C be a nonempty convex subset of complete CAT(0) space X. Let $T\colon C \to C$ and $S\colon C \to C$ be mappings satisfying condition (E) with $F \neq \emptyset$. Let $\{x_n\}$ be an iteration process described by* (2). *If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are such that $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$, for some $a, b \in (0,1)$, then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\lim_{n\to\infty} d(x_n, Sx_n) = 0$.*

Theorem 12. *Let C be a nonempty closed convex subset of a CAT(0) space X. Let $T\colon C \to C$ and $S\colon C \to C$ be mappings satisfying condition (E) with $F \neq \emptyset$. Let $\{x_n\}$ be an iteration process described by* (2). *If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are such that $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$, for some $a, b \in (0,1)$, then $\{x_n\}$ Δ - converges to a common fixed point p of T and S.*

Corollary 1. *Let C be a nonempty closed convex subset of a CAT(0) space X. Let $T\colon C \to C$ be a mapping satisfying condition (E) with $F(T) \neq \emptyset$. Let $\{x_n\}$ be an iteration process described by* (2). *If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are such that $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$, for some $a, b \in (0,1)$, then $\{x_n\}$ Δ-converges to a fixed point of T.*

Theorem 13. *Let C be a nonempty bounded closed convex subset of complete CAT(0) space X. Let $T\colon C \to C$ and $S\colon C \to C$ be mappings satisfying condition (E) with*

$F \neq \emptyset$. Let $\{x_n\}$ be an iteration process described by (2). If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are such that $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$, for some $a, b \in (0,1)$ and, in addition, the two mappings satisfy the condition (A'), then $\{x_n\}$ converges strongly to a common fixed point of T and S.

Corollary 2. *Let C be a nonempty bounded closed convex subset of complete CAT(0) space X. Let $T\colon C \to C$ be a mapping satisfying condition (E) with $F(T) \neq \emptyset$. Let $\{x_n\}$ be an iteration process described by (2). If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are such that $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$, for some $a, b \in (0,1)$ and, in addition, the mapping T satisfies the condition (A), then $\{x_n\}$ converges strongly to a fixed point of T.*

Finally, the chapter includes some numerical simulations on the Euclidean plane, as well as on the Poincaré half-plane. The section containing Example 7, Example 8 and Example 9 provides illustrative applications -with computer simulation- of mappings satisfying generalized nonexpansivity conditions in non-positively curved spaces.

Example 7. It is well known that pre-Hilbertian structures (in particular the real axis $\mathbb{R}$, or the real two-dimensional plane $\mathbb{R}^2$) are also $CAT(0)$ spaces. For start, let us consider on $\mathbb{R}^2$ the closed and bounded domain $[0,1]^2$, where we define two self-mappings as:

$$T\colon [0,1]^2 \to [0,1]^2, \quad Tx = (T_1(x_1), T_2(x_2)),$$

where

$$T_1 x = \begin{cases} \dfrac{1}{x^2+4}, \ x \neq 1; \\ \\ \dfrac{3}{5}, \ x = 1; \end{cases}, \quad T_2 x = \begin{cases} \dfrac{\ln(1+x)}{2}, \ x \neq 1; \\ \\ \dfrac{3}{5}, \ x = 1; \end{cases}$$

and

$$S\colon [0,1]^2 \to [0,1]^2, \quad Sx = (S_1(x_1), S_2(x_2)),$$

where

$$S_1 x = \begin{cases} \dfrac{1-x^3}{4}, \ x \neq 1; \\ \\ \dfrac{4}{7}, \ x = 1. \end{cases}, \quad S_2 x = \begin{cases} \dfrac{\sin x}{2}, \ x \neq 1; \\ \\ \dfrac{4}{7}, \ x = 1. \end{cases}$$

In the first part of the study, the operators T and S are proven to satisfy conditions $\left(E_{\frac{5}{2}}\right)$ and $\left(E_{\frac{7}{3}}\right)$, respectively, relative to the Euclidean norm on $\mathbb{R}^2$. Secondly, for the two mappings included in this example, we apply the iteration procedure (2) to obtain approximate value for the common fixed point and the number of executions of the iterative procedure.

Example 8. For a given real number p, consider the set $C = \{(p, x) \,:\, x > 0\} \subset \mathbf{H}$, the Poincaré half-plane. We define the mapping

$$T\colon C \to C, \quad T(p, x) = \left(p, \frac{1}{x}\right).$$

Then, T is proven to be nonexpansive on C with respect to the Poincaré metric d. However, T fails to be nonexpansive with respect to the Euclidean metric on $\mathbb{R}^2$.

Example 9. Consider the Poincaré half-plane $\mathbf{H} = \{x = (x_1, x_2) \,:\, x_1, x_2 \in \mathbb{R},\ x_2 > 0\}$, with the Poincaré metric

$$d(x, y) = 2 \ln \left(\frac{\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} + \sqrt{(y_1 - x_1)^2 + (y_2 + x_2)^2}}{2\sqrt{x_2 y_2}} \right).$$

We define the mapping $T\colon \mathbf{H} \to \mathbf{H}, \quad T(x_1, x_2) = (-x_1, x_2)$, which is nonexpansive in connection with the hyperbolic distance d.

The aim of this example is to apply the iterative procedure (2), for a selected initial estimation $x^0 \in \mathbf{H}$ in order to obtain the corresponding fixed point of T. This requires first to have a precise image about how $z = (1-t)x \oplus ty$ could be computed exactly. By means of polar coordinates and using general facts about the geodesics of the Poincaré half-plane, we were able to find:

$$(1-t)x \oplus ty = \begin{cases} (p, x_2^{1-t} y_2^t), & \text{if } x_1 = y_1 = p; \\[2ex] \left(a + R\dfrac{1 - \lambda^2(x,y,t)}{1 + \lambda^2(x,y,t)}, R\dfrac{2\lambda(x,y,t)}{1 + \lambda^2(x,y,t)}\right), & \text{if } x_1 \neq y_1, \end{cases}$$

where a, R and $\lambda(x, y, t)$ are defined below:

$$a = \frac{(y_1^2 + y_2^2) - (x_1^2 + x_2^2)}{2(y_1 - x_1)};$$

and

$$R = \frac{\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \cdot \sqrt{(y_1 - x_1)^2 + (y_2 + x_2)^2}}{2|x_1 - y_1|}.$$

$$\lambda(x, y, t) = \left(\frac{R + a - x_1}{x_2}\right)^{1-t} \left(\frac{R + a - y_1}{y_2}\right)^{t}.$$

By including this formula into the procedure (2) and running the algorithm with a given initial estimate we have found the approximate solution and the number of iterations to be performed.

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