University "Politehnica" of Bucharest
Department of Mathematics & Informatics
Splaiul Independenţei 313
060042 Bucharest, Romania

Iterated function systems with orbital contractivity conditions

PhD student Maria - Irina SAVU
E-mail address: maria_irina.savu@upb.ro

Abstract of the PhD thesis

Supervised by Lect. dr. habil. Alexandru MIHAIL

Bucharest, 2023

Contents

Motivation and outlines **6**

1. Preliminaries **17**

2. φ-Contractive parent-child possibly infinite iterated function systems and orbital φ-contractive possibly infinite iterated function systems **29**

2.1 φ-Contractive parent-child possibly infinite iterated function systems (pcIIFSs) 30

2.2 Orbital φ-contractive possibly infinite iterated function systems (oIIFSs) 46

2.3 The H_S operator 53

2.4 The Markov operator 61

2.5 Examples 65

3. On the connectedness of attractors of orbital contractive iterated function systems **70**

3.1 Preliminaries 70

3.2 A necessary and sufficient condition for an attractor to be connected 73

3.3 Arcwise connected attractors 76

3.4 Final remarks 83

4. φ-Contractive orbital affine iterated function systems **84**

4.1 Preliminaries 84

4.2 Main results 86

4.3 Remarks and examples 92

5. Orbital fuzzy iterated function systems **96**

5.1 Preliminaries 96

5.2 Results regarding the fuzzy operator associated with an orbital fuzzy iterated function system 102

5.3 A characterization of the fuzzy fractals generated by an orbital fuzzy iterated function system 118

5.4 The structure of fuzzy fractals generated by an orbital fuzzy iterated function system 127

Conclusions **133**

Further development **134**

References **136**

Key words and phrases: possibly infinite iterated function system, orbital contractive iterated function system, fractal operator, Picard operator, weakly Picard operator, attractors, canonical projection, connectedness, arcwise connectedness, orbital affine iterated function system, orbital fuzzy iterated function system, fuzzy fractal, fuzzy Hutchinson-Barnsley operator.

2020 Mathematics Subject Classification: 03E72, 28A80, 37C25, 37C70, 54H20, 54H25

Motivation and outlines

General objective. The aim of this thesis is to develop a new direction of generalization for the concept of iterated function system, which is one of the most important methods of obtaining fractals. The concept of "fractal" has not a formal definition, but it can be seen as an irregular, broken geometric shape, in some cases having the property that at any scales, every part is (at least approximately) a smaller copy of the initial figure.

One of the most important methods of obtaining fractals consists of applying fixed point theorems to a certain operator (called the fractal or the Hutchinson-Barnsley operator) associated with an iterated function system. The notion of iterated function system (denoted for short by IFS) was considered for the first time by J. Hutchinson in 1981 (see [7]) and it became more popular in 1998 when M. F. Barnsley published the book "Fractals everywhere" (see [1]).

There exists various generalizations for the concept of iterated function system. A direction of generalization is represented by considering weaker contractivity conditions for the functions of the system (see [4], [23], [24], [26]). Another direction of generalization consists of considering systems with an arbitrary number (finite or infinite) of functions (see [5], [6], [8], [13], [27]). Another way to generalize the IFSs is to change the structure of component functions or the structure of space (see [2], [3], [10], [11], [28], [29], [30]). For all the above mentioned generalizations of IFSs, the fractal operator associated with such a system is Picard and its unique fixed point is the attractor of the system.

This thesis is dedicated to the study of IFSs for which the fractal operator is weakly Picard. More precisely, we introduce and study a new class of iterated function systems, namely possibly infinite iterated function systems (denoted for short as IIFSs) for which the component functions are endowed with weaker contractivity conditions on the orbit of the elements from the space where they are defined.

The motivation of the study presented in this thesis is given by the fact that one of the most important methods of obtaining fractals is based on the notion of iterated function system. Consequently, in the last years, many mathematicians have been interested in finding a large variety of generalizations for the concept of IFS. In this thesis we define and study a new class of IFSs which are endowed with orbital contractivity conditions and applying a weaker fixed point theorem, we prove that the associated fractal operator is weakly Picard. Therefore, the IFSs presented in this thesis have a family of attractors, instead of a unique attractor. We made a first study in this direction in the paper [14], where we proved that the fractal operator associated with an IFS consisting of continuous functions satisfying Banach's orbital condition is weakly Picard. Also, inspired by the studies regarding the infinite iterated function systems, the IFSs with weaker contractivity conditions and the fuzzy IFSs presented in [5], [6], [12] and [23], we took into consideration the idea of combining these concepts with the class of IFSs introduced in this thesis and for these new obtained systems we studied the associated operators.

Thesis description. The aim of this PhD thesis is to develop a new direction of generalization for the concept of iterated function system. In Chapter 1 of this thesis we introduce the notions of φ-contractive parent-child possibly infinite iterated function system (pcIIFS) and orbital φ-contractive possibly infinite iterated function system (oIIFS). Also, in this Chapter, we state the notations and terminology that we used in the thesis. In Chapter 2, we prove that the fractal operator associated with a pcIIFS or an oIIFS is weakly Picard. For these types of systems we construct the canonical projection and we study its properties. Moreover, the operator H_S which was studied for φ-contractions is generalized for pcIIFSs and oIIFSs and we prove that the generalized operator is continuous and weakly Picard. Also, we use this generalization of the H_S operator to prove that the Markov operator associated with a finite pcIIFS or oIIFS with probabilities is weakly Picard. In Chapter 3, we study certain topological properties of attractors of orbital contractive iterated function systems (which are a particular case of oIIFSs, where there is a finite number of functions in the system and the comparison function is linear). In Chapter 4, we introduce the notion of φ-contractive orbital affine iterated function system (oAIFS) and we present two structure results for the component functions of an oAIFS. In Chapter 5, we introduce the notion of orbital fuzzy iterated function system and we prove that the associated fuzzy operator is weakly Picard. Moreover, we present some structure results regarding the fixed points of the fuzzy operator. In every chapter some examples are provided.

Chapter 1

Preliminaries

In this chapter we state the notations and terminology used in this thesis.

Definition 1.1. Let (X, d) be a metric space.

1) A weakly Picard operator is a function $f: X \to X$ having the property that for every $x \in X$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ is convergent to a fixed point of f. In this case, we define the operator $f^\infty: X \to X$ given by $f^\infty(x) = \lim_{n\to\infty} f^n(x)$ for every $x \in X$.

2) A Picard operator is a weakly Picard operator which has a unique fixed point.

Definition 1.2. A function $\varphi: [0, \infty) \to [0, \infty)$ is called

1) comparison function if $\varphi(r) < r$ for all $r > 0$ and φ is increasing;

2) summable comparison function if φ is a comparison function and $\sum_{n=0}^{\infty} \varphi^n(r)$ is convergent for every $r > 0$.

Definition 1.3. Let (X, d) be a complete metric space. A function $f: X \to X$ is called φ-contraction if there exists $\varphi: [0, \infty) \to [0, \infty)$ a right continuous comparison function such that $d(f(x), f(y)) \leq \varphi(d(x, y))$ for every $x, y \in X$.

Let (X, d) be a metric space. In the sequel, we shall use the following notations:

$$P_b(X) = \{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is bounded}\};$$

$$P_{cl,b}(X) = \{A \in P_b(X) \mid A \text{ is closed}\};$$

$$P_{cp}(X) = \{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is compact}\}.$$

For $A \subseteq X$, the diameter of the set A is $\operatorname{diam}(A) = \sup_{x,y\in A} d(x, y)$ and for a fixed $x \in X$, the distance between x and A is $d(x, A) = \inf_{y\in A} d(x, y)$.

For two functions $f, g: X \to X$, the uniform distance between f and g is $d_u(f, g) = \sup_{x\in X} d(f(x), g(x))$.

On $P_b(X)$ we define the generalized Hausdorff-Pompeiu semidistance $h\colon P_b(X)\times P_b(X)\to[0,\infty)$ given by $h(A,B)=\max\{d(A,B),d(B,A)\}$ for all $A,B\in P_b(X)$, where $d(A,B)=\sup\limits_{x\in A}d(x,B)$. The restriction of h to $P_{cl,b}(X)$ is called the Hausdorff-Pompeiu metric and it is also denoted by h.

The shift space

For $n\in\mathbb{N}^*$, by $\Lambda_n(I)$ we mean the set of all finite words with n letters of I, namely $\omega=\omega_1\omega_2\cdots\omega_n$. In this case, n is called the length of ω and it is denoted by $|\omega|$. For $\omega\in\Lambda_n(I)$ and $m\in\mathbb{N}^*$, by $[\omega]_m$ we mean the word formed with the first m letters of ω if $m\le n$, or the word ω if $n<m$. By $[\omega]_0$ we mean the word λ. $\Lambda_0(I)=\{\lambda\}$, where λ is called the empty word.

For a set I, by $\Lambda(I)$ we mean the set of infinite words, namely $\omega=\omega_1\omega_2\cdots\omega_n\cdots$, where $\omega_1,\ \omega_2,\ \cdots\ \omega_n,\ \cdots\in I$. For $\omega\in\Lambda(I)$ and $n\in\mathbb{N}^*$ by $[\omega]_n$ we mean the word formed with the first n letters of ω. By $[\omega]_0$ we mean λ.

For $\alpha\in\Lambda_n(I)$ and $\beta\in\Lambda_m(I)$ or $\beta\in\Lambda(I)$, with $m,n\in\mathbb{N}^*$, by $\alpha\beta$ we mean the concatenation of α and β.

For a family $(f_i)_{i\in I}$, where $f_i\colon X\to X$ for every $i\in I$, $n\in\mathbb{N}^*$ and $\omega=\omega_1\omega_2\cdots\omega_n\in\Lambda_n(I)$, we use the notation $f_\omega=f_{\omega_1}\circ\cdots\circ f_{\omega_n}$. By f_λ we mean the identity function.

For a set $B\subseteq X$, $n\in\mathbb{N}^*$ and $\omega\in\Lambda_n(I)$, we use the notation $B_\omega=f_\omega(B)$.

By $\Lambda^*(I)$ we mean the set of all finite words, namely $\Lambda^*(I)=\bigcup\limits_{n\in\mathbb{N}}\Lambda_n(I)$.

By $\Lambda^t(I)$ we mean the set of all words with letters of I, namely $\Lambda^*(I)\cup\Lambda(I)$.

Definition 1.4. Let (X,d) and (Y,ρ) be two metric spaces and $(f_i)_{i\in I}$ a family of functions with $f_i\colon X\to Y$ for every $i\in I$. The family $(f_i)_{i\in I}$ is called

1) bounded if the set $\bigcup\limits_{i\in I}f_i(B)\in P_b(X)$ for every $B\in P_b(X)$,

2) equi-uniformly continuous if for every $\varepsilon>0$ there exists $\delta_\varepsilon>0$ such that for all $x,y\in X$ with $d(x,y)<\delta_\varepsilon$ we have $\rho(f_i(x),f_i(y))<\varepsilon$, for all $i\in I$.

Let (X,d) be a complete metric space and $(f_i)_{i\in I}$ a family of continuous functions, with $f_i\colon X\to X$ for all $i\in I$. Let $B\in P_b(X)$. By the orbit of B we mean the set $\mathcal{O}(B)=\bigcup\limits_{n\in\mathbb{N}}\overline{\bigcup\limits_{\alpha\in\Lambda(I)}f_{[\alpha]_n}(B)}$. If $B=\{x\}$, the orbit of $\{x\}$ is denoted by $\mathcal{O}(x)$.

Definition 1.5. Let (X,d) be a complete metric space and $(f_i)_{i\in I}$ a family of functions, where $f_i\colon X\to X$ for every $i\in I$. The pair denoted by $\mathcal{S}=\big((X,d),(f_i)_{i\in I}\big)$ is called possibly infinite iterated function system (IIFS for short) if

i) $f_i\colon X\to X$ is a continuous function for every $i\in I$,

ii) the family $(f_i)_{i\in I}$ is equi-uniformly continuous on bounded sets, i.e. for every $B \in P_b(X)$ and every $\varepsilon > 0$ there exists $\delta_{\varepsilon,B} > 0$ such that for all $x, y \in B$ with $d(x,y) < \delta_{\varepsilon,B}$ we have $d(f_i(x), f_i(y)) < \varepsilon$, for all $i \in I$,

iii) $(f_i)_{i\in I}$ is a bounded family of functions.

Given an IIFS $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$, the function $F_{\mathcal{S}}: P_b(X) \to P_{cl,b}(X)$, defined by

$$F_{\mathcal{S}}(B) = \overline{\bigcup_{i\in I} f_i(B)}$$

for every $B \in P_b(X)$, is called the fractal operator associated with $\mathcal{S}$.

Definition 1.5 iii) ensures us that $F_{\mathcal{S}}(B) \in P_{cl,b}(X)$ for every $B \in P_b(X)$.

The restriction of $F_{\mathcal{S}}$ to $P_{cl,b}(X)$ will still be denoted by $F_{\mathcal{S}}$.

Definition 1.6. Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an IIFS and let $F_{\mathcal{S}}: P_{cl,b}(X) \to P_{cl,b}(X)$ be the fractal operator associated with $\mathcal{S}$. Every fixed point of $F_{\mathcal{S}}$ is called an attractor of $\mathcal{S}$. We say that $\mathcal{S}$ has a unique attractor if there exists a unique set denoted by $A \in P_{cl,b}(X)$ such that $F_S(A) = A$ and $\lim_{n\to\infty} h(F_S^n(K), A) = 0$ for every $K \in P_{cl,b}(X)$.

Definition 1.7. An IIFS $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ is called

1) φ-contractive parent-child possibly infinite iterated function system (pcIIFS for short) if there exists $\varphi: [0,\infty) \to [0,\infty)$ a summable comparison function such that

$$d(f_\omega(x), f_{\omega i}(x)) \le \varphi^{|\omega|}(d(x, f_i(x))),$$

for every $i \in I$, $\omega \in \Lambda^*(I)$ and $x \in X$;

2) orbital φ-contractive possibly infinite iterated function system (oIIFS for short) if there exists $\varphi: [0,\infty) \to [0,\infty)$ a right-continuous comparison function such that

$$d(f_i(y), f_i(z)) \le \varphi(d(y,z))$$

for every $i \in I$, $x \in X$ and $y, z \in \mathcal{O}(x)$.

Definition 1.8. Let (X,d) be a complete metric space and $(f_i)_{i\in I}$ a finite family of continuous functions, where $f_i: X \to X$ for all $i \in I$. The pair $\left((X,d), (f_i)_{i\in I}\right) \overset{\text{not}}{=} \mathcal{S}$ is called

1) A contractive iterated function system if there exists $C \in [0,1)$ such that

$$d(f_i(x), f_i(y)) \le C \cdot d(x,y)$$

for every $i \in I$ and $x, y \in X$.

2) A φ-contractive iterated function system if there exists $\varphi\colon [0,\infty) \to [0,\infty)$ a right continuous comparison function such that

$$d\left(f_i\left(x\right), f_i\left(y\right)\right) \leq \varphi\left(d\left(x, y\right)\right)$$

for every $i \in I$ and $x, y \in X$.

3) A parent-child contractive iterated function system if there exists $C \in [0,1)$ such that

$$d\left(f_\omega\left(x\right), f_{\omega i}\left(x\right)\right) \leq C^{|\omega|} \cdot d\left(x, f_i\left(x\right)\right)$$

for every $x \in X$, $i \in I$ and $\omega \in \Lambda^*\left(I\right)$.

4) An orbital contractive iterated function system if there exists $C \in [0,1)$ such that

$$d\left(f_i\left(y\right), f_i\left(z\right)\right) \leq C \cdot d\left(y, z\right)$$

for every $x \in X$, $i \in I$ and $y, z \in \mathcal{O}\left(x\right)$.

5) A φ-contractive orbital iterated function system if there exists $\varphi\colon [0,\infty) \to [0,\infty)$ a right continuous comparison function such that

$$d\left(f_i\left(y\right), f_i\left(z\right)\right) \leq \varphi\left(d\left(y, z\right)\right)$$

for every $x \in X$, $i \in I$ and $y, z \in \mathcal{O}\left(x\right)$.

The fractal operator associated with a system $\mathcal{S} = \left(\left(X, d\right), \left(f_i\right)_{i\in I}\right)$ from Definition 1.8, consisting of a finite number of functions, is $F_{\mathcal{S}}\colon P_{cp}\left(X\right) \to P_{cp}\left(X\right)$ given by

$$F_{\mathcal{S}}\left(K\right) = \bigcup_{i\in I} f_i\left(K\right)$$

for every $K \in P_{cp}\left(X\right)$.

Chapter 2

φ-Contractive parent-child possibly infinite iterated function systems and orbital φ-contractive possibly infinite iterated function systems

In this chapter, we study the fractal operator associated with a φ-contractive parent-child possibly infinite iterated function system (pcIIFS) and we prove that it is weakly Picard. Moreover, we present the properties of the canonical projection associated with a pcIIFS. Also, for an orbital φ-contractive possibly infinite iterated function system (oIIFS) we study the equivalent properties with the ones for pcIIFSs and the canonical projection. Related to these two types of systems, we generalize the H_S operator presented in [17] for φ-contractions and we use this operator to prove that the Markov operator associated with a finite pcIIFS or oIIFS with probabilities is weakly Picard. In the last part, we give some examples. The results presented here were published in [18] and [19].

2.1 φ-Contractive parent-child possibly infinite iterated function systems (pcIIFSs)

Let (X, d) be a complete metric space.

Theorem 2.1 (see [18]). *Let $\mathcal{S} = \left((X, d), (f_i)_{i\in I}\right)$ be a pcIIFS and let $F_{\mathcal{S}}\colon P_{cl,b}(X) \to P_{cl,b}(X)$ be the fractal operator associated with $\mathcal{S}$. Then, $F_{\mathcal{S}}$ is a weakly Picard operator. More precisely, for every $B \in P_{cl,b}(X)$ there exists $A_B \in P_{cl,b}(X)$ such that*

$\lim_{n\to\infty} F^n_{\mathcal{S}}(B) = A_B$ *and* $F_{\mathcal{S}}(A_B) = A_B$. *Moreover,*

$$h\left(F^n_{\mathcal{S}}(B), A_B\right) \leq \sum_{k\geq n} \varphi^k\left(\operatorname{diam}\left(B \cup F_{\mathcal{S}}(B)\right)\right)$$

for all $n \in \mathbb{N}$.

Corollary 2.1 (see [14]). *Every parent-child contractive iterated function system has attractor. More precisely, the associated fractal operator is weakly Picard.*

Proposition 2.1 (see [18]). *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be a pcIIFS. Then,* $A_B = \overline{\bigcup_{x\in B} A_x}$ *for every* $B \in P_{cl,b}(X)$.

Theorem 2.2 (see [18]). *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be a pcIIFS. Then the function* $F^\infty_{\mathcal{S}} \colon P_{cl,b}(X) \to P_{cl,b}(X)$, *given by* $F^\infty_{\mathcal{S}}(B) = \lim_{m\to\infty} F^m_{\mathcal{S}}(B)$, *for every* $B \in P_{cl,b}(X)$, *is continuous.*

Proposition 2.2 (see [18]). *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be a pcIIFS. Then, for every* $B \in P_b(X)$ *and* $\alpha \in \Lambda(I)$, *the sequence* $\left(\overline{f_{[\alpha]_n}(B)}\right)_{n\in\mathbb{N}}$ *is convergent. If we denote by* $a_\alpha(B) = \lim_{n\to\infty} \overline{f_{[\alpha]_n}(B)}$, *then* $h\left(f_{[\alpha]_m}(B), a_\alpha(B)\right) \leq \sum_{k=m}^{\infty} \varphi^k\left(\operatorname{diam}\left(\mathcal{O}(B)\right)\right)$ *for all* $m \in \mathbb{N}$.

Lemma 2.1 (see [18]). *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be a pcIIFS. Then,*
1) $a_\alpha(B) = \overline{\bigcup_{x\in B} \{a_\alpha(x)\}}$ *for every* $B \in P_b(X)$ *and* $\alpha \in \Lambda(I)$;
2) a_α *is uniformly continuous on* B *for every* $\alpha \in \Lambda(I)$ *and* $B \in P_{cl,b}(X)$;
3) $f_\omega(a_\alpha(B)) = a_{\omega\alpha}(B)$ *for every* $\alpha \in \Lambda(I)$, $\omega \in \Lambda_n(I)$, $n \in \mathbb{N}^*$ *and* $B \in P_b(X)$;
4) $a_\alpha(x) = a_\alpha(y)$ *for every* $x \in X$, $y \in \overline{\mathcal{O}(x)}$ *and* $\alpha \in \Lambda(I)$.

Theorem 2.3 (see [18]). *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be a pcIIFS. Then,* $A_B = \overline{\bigcup_{\alpha\in\Lambda(I)} a_\alpha(B)} = \overline{\bigcup_{x\in B} \bigcup_{\alpha\in\Lambda(I)} \{a_\alpha(x)\}}$ *for every* $B \in P_{cl,b}(X)$.

Proposition 2.3 (see [18]). *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be a pcIIFS. Then,* $A_B = A_x$ *for every* $x \in X$ *and* $B \in P_{cl,b}\left(\overline{\mathcal{O}(x)}\right)$.

Proposition 2.4 (see [18]). *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be a pcIIFS. Then, for every* $x, y \in X$ *such that* $\overline{\mathcal{O}(x)} \cap \overline{\mathcal{O}(y)} \neq \emptyset$, *we have* $A_x = A_y$. *In particular, if* $\overline{\mathcal{O}(x)} \cap \overline{\mathcal{O}(y)} \neq \emptyset$ *for all* $x, y \in X$, *we have that* $F_{\mathcal{S}}$ *is a Picard operator.*

Proposition 2.5 (see [18]). *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be a pcIIFS. Then,* $\left(\operatorname{diam}\left(A_{[\alpha]_n,x}\right)\right)_{n\in\mathbb{N}}$ *is convergent to* 0 *and* $\{a_\alpha(x)\} = \lim_{n\to\infty} A_{[\alpha]_n,x}$ *for every* $x \in X$ *and* $\alpha \in \Lambda(I)$.

Theorem 2.4 (see [18]). *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be a pcIIFS. Then the function* $\Theta\colon \Lambda^t(I) \times P_{cl,b}(X) \to P_{cl,b}(X)$ *defined by*

$$\Theta(\alpha, B) = \begin{cases} a_\alpha(B), \text{ if } \alpha \in \Lambda(I) \\ f_\alpha(B), \text{ if } \alpha \in \Lambda^*(I) \setminus \{\lambda\} \\ B, \text{ if } \alpha = \lambda \end{cases}$$

for all $(\alpha, B) \in \Lambda^t(I) \times P_{cl,b}(X)$ *is uniformly continuous on bounded sets. In particular,* Θ *is continuous.*

Theorem 2.5 (see [25]). *Let* $\mathcal{S}_1 = \left((X,d), (f_i)_{i\in\{1,\cdots,n\}}\right)$ *and* $\mathcal{S}_2 = \left((X,d), (g_i)_{i\in\{1,\cdots,n\}}\right)$ *be two parent-child contractive iterated function systems, such that:*

i) there exists $C \in (0,1)$ *such that* $F_{\mathcal{S}_1}\colon \mathcal{O}_{\mathcal{S}_1}(x) \to \mathcal{O}_{\mathcal{S}_1}(x)$ *and* $F_{\mathcal{S}_2}\colon \mathcal{O}_{\mathcal{S}_2}(x) \to \mathcal{O}_{\mathcal{S}_2}(x)$ *are Banach contractions with the contraction constant* C*, for every* $x \in X$*, where* $\mathcal{O}_{\mathcal{S}_1}(x) = \{f_{\alpha_1\cdots\alpha_m}(x) \mid m \in \mathbb{N}, \alpha_1, \cdots, \alpha_m \in \{1, \cdots, n\}\}$ *and* $\mathcal{O}_{\mathcal{S}_2}(x) = \{g_{\alpha_1\cdots\alpha_m}(x) \mid m \in \mathbb{N}, \alpha_1, \cdots, \alpha_m \in \{1, \cdots, n\}\}$.

ii) $\mathcal{S} = \left((X,d), (l_i)_{i\in\{1,\cdots,2n\}}\right)$ *where* $l_i = f_i$ *and* $l_{n+i} = g_i$ *for every* $i \in \{1, \cdots, n\}$*, is a parent-child contractive iterated function system. Then*

$$h(FixF_{\mathcal{S}_1},\ FixF_{\mathcal{S}_2}) \leq \frac{1}{1-C} \max_{i=1}^{n} d_u(f_i,\ g_i),$$

where by $FixF_{\mathcal{S}}$ *we mean the set of fixed points of the fractal operator* $F_{\mathcal{S}}$.

2.2 Orbital φ-contractive possibly infinite iterated function systems (oIIFSs)

Theorem 2.6 (see [18]). *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be an oIIFS and let* $F_{\mathcal{S}}\colon P_{cl,b}(X) \to P_{cl,b}(X)$ *be the fractal operator associated with* $\mathcal{S}$*. Then,* $F_{\mathcal{S}}$ *is a weakly Picard operator.*

Moreover, $h(F^n_{\mathcal{S}}(B), A_B) \leq \varphi^n\left(\operatorname{diam}\left(\overline{\mathcal{O}(B)}\right)\right)$ *for all* $n \in \mathbb{N}$ *and* $B \in P_{cl,b}(X)$*, where* $A_B = \overline{\bigcup_{x\in B} A_x}$.

Theorem 2.7 (see [18]). *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be an oIIFS. Then the function* $F^\infty_{\mathcal{S}}\colon P_{cl,b}(X) \to P_{cl,b}(X)$*, given by* $F^\infty_{\mathcal{S}}(B) = \lim_{m\to\infty} F^m_{\mathcal{S}}(B)$ *for all* $B \in P_{cl,b}(X)$*, is continuous.*

Proposition 2.6 (see [18]). *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be an oIIFS. Then, for all* $B \in P_b(X)$ *and* $\alpha \in \Lambda(I)$*, the sequence* $\left(f_{[\alpha]_n}(B)\right)_{n\in\mathbb{N}}$ *is convergent. If we denote its limit* *by* $a_\alpha(B)$*, we have* $h\left(f_{[\alpha]_n}(B), a_\alpha(B)\right) \leq \varphi^n(\text{diam}(\mathcal{O}(B)))$ *for all* $n \in \mathbb{N}$*. Moreover,* $a_\alpha(B) = \overline{\bigcup_{x \in B} \{a_\alpha(x)\}}$.

Theorem 2.8 (see [18])**.** *Let* $\mathcal{S} = \left((X,d), (f_i)_{i \in I}\right)$ *be an oIIFS. Then,* $A_B = \overline{\bigcup_{\alpha \in \Lambda(I)} a_\alpha(B)} = \overline{\bigcup_{x \in B} \bigcup_{\alpha \in \Lambda(I)} \{a_\alpha(x)\}}$ *for every* $B \in P_{cl,b}(X)$.

Theorem 2.9 (see [18])**.** *Let* $\mathcal{S} = \left((X,d), (f_i)_{i \in I}\right)$ *be an oIIFS. Then the function* $\Theta \colon \Lambda^t(I) \times P_{cl,b}(X) \to P_{cl,b}(X)$ *defined by*

$$\Theta(\alpha, B) = \begin{cases} a_\alpha(B), \text{ if } \alpha \in \Lambda(I) \\ f_\alpha(B), \text{ if } \alpha \in \Lambda^*(I) \setminus \{\lambda\} \\ B, \text{ if } \alpha = \lambda \end{cases}$$

for all $(\alpha, B) \in \Lambda^t(I) \times P_{cl,b}(X)$ *is uniformly continuous on bounded sets.*

2.3 The $H_{\mathcal{S}}$ operator

Given (X,d) and (Y,ρ) two metric spaces, we use the following notations:
$\mathcal{C}(X,Y) = \{f \colon X \to Y \mid f \text{ is continuous}\}$; $\mathcal{C}_b(X,Y) = \{f \in \mathcal{C}(X,Y) \mid f \text{ is bounded}\}$.

Let $\mathcal{S} = \left((X,d), (f_i)_{i \in I}\right)$ be an IIFS. By $\widetilde{\mathcal{C}}$ we mean the set $\widetilde{\mathcal{C}} = \{(f,g) \mid f \in \mathcal{C}_b(Y \times \Lambda(I), X),\ g \in \mathcal{C}_b(Y,X),$ such that $f(y,\alpha) \in \overline{\mathcal{O}(g(y))}$ for all $y \in Y, \alpha \in \Lambda(I)\}$.

On $\mathcal{C}_b(Y \times \Lambda(I), X)$ we define the operator $H_{\mathcal{S}} \colon \mathcal{C}_b(Y \times \Lambda(I), X) \to \mathcal{C}_b(Y \times \Lambda(I), X)$ given by $H_{\mathcal{S}}(f)(y, i\alpha) = f_i \circ f(y,\alpha)$ for all $f \in \mathcal{C}_b(Y \times \Lambda(I), X)$, $y \in Y$, $i \in I$ and $\alpha \in \Lambda(I)$.

On $\widetilde{\mathcal{C}}$ we consider $\widetilde{H}_{\mathcal{S}} \colon \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$ defined by $\widetilde{H}_{\mathcal{S}}(f,g) = (H_{\mathcal{S}}(f), g)$ for all $(f,g) \in \widetilde{\mathcal{C}}$.

Proposition 2.7. *Let* $\mathcal{S} = \left((X,d), (f_i)_{i \in I}\right)$ *be an IIFS. Then* $\widetilde{H}_{\mathcal{S}}$ *is continuous and* $\widetilde{H}_{\mathcal{S}}\left(\widetilde{\mathcal{C}}\right) \subset \widetilde{\mathcal{C}}$*. Moreover, for every* $(f,g) \in \widetilde{\mathcal{C}}$*, the sequence* $\left(\widetilde{H}^n_{\mathcal{S}}(f,g)\right)_n$ *is convergent.*

For $g \in \mathcal{C}_b(Y,X)$ we shall use the notation $\widetilde{\mathcal{C}}_g = \{(f,g) \mid f \in \mathcal{C}_b(Y \times \Lambda(I), X),$ $f(y,\alpha) \in \overline{\mathcal{O}(g(y))}$ for all $y \in Y,\ \alpha \in \Lambda(I)\}$.

Theorem 2.10. *Let* $\mathcal{S} = \left((X,d), (f_i)_{i \in I}\right)$ *be an oIIFS. Then the function* $\widetilde{H}_{\mathcal{S}} \colon \widetilde{\mathcal{C}}_g \to \widetilde{\mathcal{C}}_g$ *is a Picard operator.*

Theorem 2.11. *Let* $\mathcal{S} = \left((X,d), (f_i)_{i \in I}\right)$ *be a pcIIFS or an oIIFS. Then, the operator* $\widetilde{H}_{\mathcal{S}} \colon \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$ *is weakly Picard.*

2.4 The Markov operator

By $\mathcal{B}(X)$ we mean the σ-algebra of Borel subsets of X. By $\mathcal{M}(X)$ we mean the set of all borelian probability measures $\mu: \mathcal{B}(X) \to [0,\infty)$ which have compact support.

Definition 2.1. Let $\left((X,d),(f_i)_{i\in I}\right)$ be a pcIIFS which contains a finite number of functions or an oIIFS formed with a finite number of functions and let $(p_i)_{i\in I}$ be a finite family, where $p_i \in (0,1)$ for all $i \in I$ and $\sum\limits_{i\in I} p_i = 1$. By an iterated function system with probabilities (IFSp for short) we mean the system denoted by $\mathcal{S} = \left((X,d),(f_i)_{i\in I},(p_i)_{i\in I}\right)$.

Definition 2.2. Given an IFSp $\mathcal{S} = \left((X,d),(f_i)_{i\in I},(p_i)_{i\in I}\right)$, we define an operator $M_{\mathcal{S}}: \mathcal{M}(X) \to \mathcal{M}(X)$ called the Markov operator associated with $\mathcal{S}$, given by $M_{\mathcal{S}}(\mu) = \sum\limits_{i\in I} p_i \cdot \mu \circ f_i^{-1}$ for all $\mu \in \mathcal{M}(X)$.

The following result shows that the Markov operator associated with an IFSp $\mathcal{S}$ is a weakly Picard operator and the proof is based on the properties of $H_{\mathcal{S}}$ operator associated with the system $\mathcal{S}$.

Theorem 2.12. *Let $\mathcal{S} = \left((X,d),(f_i)_{i\in I},(p_i)_{i\in I}\right)$ be an IFSp and let $M_{\mathcal{S}}: \mathcal{M}(X) \to \mathcal{M}(X)$ be the Markov operator associated with $\mathcal{S}$. Then $M_{\mathcal{S}}$ is a weakly Picard operator.*

2.5 Examples

By π_1 we mean the function $\pi_1: X \times Y \to X$ given by $\pi_1(x,y) = x$ for all $(x,y) \in X \times Y$ and by π_2 we mean the function $\pi_2: X \times Y \to Y$ given by $\pi_2(x,y) = y$ for all $(x,y) \in X \times Y$.

Example A. Let us consider the metric space $(\mathbb{R}^2, \|\cdot\|_2)$, where $\|\cdot\|_2$ is the euclidean norm on $\mathbb{R}^2$, and the subset $X = \{(x,y) \in \mathbb{R}^2 \mid x \geq -y^2 - 100\}$. We define the functions $f_0, f_1: X \to X$ given by $f_0(x,y) = \left(\frac{x}{3}, y\right)$ and $f_1(x,y) = \left(\frac{x}{3} + \frac{2}{3}y^2, y\right)$ for all $(x,y) \in X$. Thus, $f_i(x,y) = \left(\frac{x}{3} + \frac{2i}{3}y^2, y\right)$ for all $i \in \{0,1\}$ and $(x,y) \in X$.

We proved that $\mathcal{S} = \left((X,d),(f_i)_{i\in\{0,1\}}\right)$ is a pcIIFS.

Let $\alpha = i_1 i_2 \cdots i_n \cdots \in \Lambda(I)$. We have $a_\alpha(x,y) = \left(2\sum\limits_{n\geq 1} \frac{i_n}{3^n} y^2, y\right)$ for all $(x,y) \in X$. Therefore, for every $(x,y) \in X$, the attractor is $A_{(x,y)} = (2Cy^2, y)$, where C is the Cantor set. For a set $B \in P_{cl,b}(X)$ we have $A_B = \{(2ty^2, y) \mid \text{there exist } t \in C \text{ and } (x,y) \in B\}$ and

$$\widetilde{H}_{\mathcal{S}}^m(f,g)(y,\beta\alpha) = \left(H_{\mathcal{S}}^m(f)(y,\beta\alpha), g(y)\right) = \left(f_\beta(f(y,\alpha)), g(y)\right)$$

$$= \left(\frac{1}{3^m} f_{[1]}(y,\alpha) + 2\left(\frac{i_1}{3} + \frac{i_2}{3^2} + \cdots + \frac{i_m}{3^m} \right) f^2_{[2]}(y,\alpha), f^2_{[2]}(y,\alpha), g(y) \right)$$

for all $m \in \mathbb{N}^*$, $\beta = i_1 i_2 \cdots i_m \in \Lambda_m(I)$, $\alpha \in \Lambda(I)$, and $(f,g) \in \widetilde{\mathcal{C}}$.

Example B. Let $\alpha \in \left(\frac{3}{4}, 1\right)$ and $\beta \in \mathbb{R}$, such that $0 < \beta < \alpha^3$. We consider the set $X \subset \mathbb{R}$ defined by $X = \{0, 1, \alpha, \alpha+\alpha^2, \alpha+\alpha^2+\alpha^3, \alpha+\alpha^2+\alpha^3-\beta\}$ and we denote by d the usual metric on $\mathbb{R}$, namely $d: \mathbb{R}\times\mathbb{R} \to [0,\infty)$, $d(x,y) = |x-y|$ for all $(x,y) \in \mathbb{R}\times\mathbb{R}$. On the metric space (X,d) we consider two functions $f_1, f_2: X \to X$ given by $f_1(1) = f_2(1) = 0$, $f_1(0) = f_2(0) = \alpha$, $f_1(\alpha) = f_2(\alpha) = \alpha+\alpha^2$, $f_1(\alpha+\alpha^2) = \alpha+\alpha^2+\alpha^3$, $f_2(\alpha+\alpha^2) = \alpha+\alpha^2+\alpha^3-\beta$, $f_1(\alpha+\alpha^2+\alpha^3) = \alpha+\alpha^2+\alpha^3$, $f_2(\alpha+\alpha^2+\alpha^3) = \alpha+\alpha^2+\alpha^3-\beta$, $f_1(\alpha+\alpha^2+\alpha^3-\beta) = \alpha+\alpha^2+\alpha^3$, $f_2(\alpha+\alpha^2+\alpha^3-\beta) = \alpha+\alpha^2+\alpha^3-\beta$.

We proved that $\mathcal{S} = \left((X,d), (f_i)_{i\in\{1,2\}}\right)$ is a pcIIFS with the summable comparison function $\varphi: [0,\infty) \to [0,\infty)$ given by $\varphi(t) = \alpha t$ for all $t \in [0,\infty)$ but $\widetilde{\mathcal{S}} = \left(\left(\widetilde{\mathcal{C}}, \widetilde{d}\right), \widetilde{H}_{\mathcal{S}}\right)$ is not a pcIIFS with respect to φ.

Example C. We consider the space $(\mathbb{R}^2, \|\cdot\|_2)$, where $\|\cdot\|_2$ is the Euclidean norm. Let $D_1 = \left\{(x,y) \in \mathbb{R}^2 |\ x > 4,\ y \leq 4 + \frac{1}{x+4}\right\}$ and $D_2 = \left\{(x,y) \in \mathbb{R}^2 |\ x < -4,\ y \leq 4 - \frac{1}{x-4}\right\}$. We consider the functions f_1, f_2, with $f_1, f_2: \mathbb{R}^2 \setminus (D_1 \cup D_2) \to \mathbb{R}^2$, given by $f_1(x,y) = \left(\frac{1}{3}x + \frac{1}{3}, y\right)$ and $f_2(x,y) = \left(\frac{2}{3}x, y\right)$ for all $(x,y) \in \mathbb{R}^2 \setminus (D_1 \cup D_2)$.

One can easily prove that $\mathcal{S} = \left((\mathbb{R}^2, \|\cdot\|_2), (f_i)_{i\in\{1,2\}}\right)$ is a pcIIFS and also an oIIFS. For an element $K \in P_{cp}(\mathbb{R}^2 \setminus (D_1 \cup D_2))$, the corresponding attractor is $A_K = \left[0, \frac{1}{2}\right] \times \pi_2(K)$.

Example D. Let us consider the metric space $(\mathbb{R}^2, \|\cdot\|_2)$, where $\|\cdot\|_2$ is the Euclidean norm and $a > 0$. We denote by $\mathcal{S}_1$ the system $\left((\mathbb{R}^2, \|\cdot\|_2), (f_i)_{i\in\{1,2\}}\right)$, where f_1, $f_2: \mathbb{R}^2 \to \mathbb{R}^2$ are given by $f_1(x,y) = \left(\frac{1}{2}x, y\right)$ and $f_2(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, y\right)$ for all $(x,y) \in \mathbb{R}^2$. For a set $K \in P_{cp}(\mathbb{R}^2)$, the attractor is $A^1_K = [0,1] \times \pi_2(K)$.

We denote by $\mathcal{S}_2$ the system $\left((\mathbb{R}^2, \|\cdot\|_2), (g_i)_{i\in\{1,2\}}\right)$, where g_1, $g_2: \mathbb{R}^2 \to \mathbb{R}^2$ are given by $g_1(x,y) = \left(\frac{1}{2}x, y\right)$ and $g_2(x,y) = \left(\frac{1}{2}x + \alpha, y\right)$ for all $(x,y) \in \mathbb{R}^2$. Given an element $K \in P_{cp}(\mathbb{R}^2)$, the corresponding attractor for $\mathcal{S}_2$ is $A^2_K = [0, 2\alpha] \times \pi_2(K)$.

One can see that $\mathcal{S}_1$ and $\mathcal{S}_2$ are both pcIIFSs and oIIFSs. For $K \in P_{cp}(\mathbb{R}^2)$, we have

$$H(FixF_{\mathcal{S}_1}, FixF_{\mathcal{S}_2}) \leq \tfrac{1}{1-\frac{1}{2}} \max_{i=1}^{2} d_u(f_i, g_i)$$

obtaining a confirmation of Theorem 2.5.

Chapter 3

On the connectedness of attractors of orbital contractive iterated function systems

In this chapter, we study certain topological properties of attractors of orbital contractive iterated function systems (which are a particular case of oIIFSs where there is a finite number of functions in the system and the comparison function is linear). We give a necessary and sufficient condition for an attractor to be connected and sufficient conditions for an attractor to be arcwise connected. We offer a generalization for the last result and we study under which conditions an attractor has a finite number of arcwise connected components. We provide some examples. These results are published in [20].

3.1 A necessary and sufficient condition for an attractor to be connected

Definition 3.1. A subset $A \neq \emptyset$ of a metric space (X, d) is said to be arcwise connected if for every $x, y \in A$, there exists a continuous map γ: $[0, 1] \to A$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Definition 3.2. Let A be a nonempty subset of a metric space (X, d) and $x, y \in A$. We consider the relation $\widetilde{R}$ on the set A given by $x\widetilde{R}y$ if there exists $B \subset A$, B arcwise connected, such that $x, y \in B$. It is easy to show that $\widetilde{R}$ is an equivalence relation. The equivalence classes of $\widetilde{R}$ are called the arcwise connected components of A.

Let $\mathcal{S} = \left((X, d), (f_i)_{i\in I}\right)$ be an orbital contractive iterated function system. For a fixed $K \in P_{cp}(X)$, let us consider the set $M_K := \{A_x \mid x \in K\}$. Let $g\colon A_K \to M_K$ be the function defined by $g(x) = A_x$ for every $x \in A_K$. As g is continuous and surjective, M_K is compact.

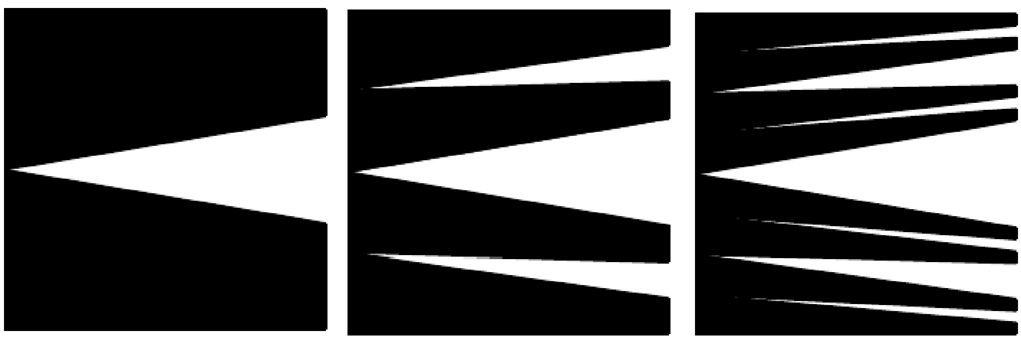
Theorem 3.1 (see [20]). *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an orbital contractive iterated function system and $K \in P_{cp}(X)$. Then, A_K is connected if and only if the family $(A_{K,i})_{i\in I}$ is connected and M_K is connected.*

3.2 Arcwise connected attractors

Example A. Let us consider the normed space $(X,d) = (\mathbb{R}^2, \|\cdot\|_2)$, where $\|\cdot\|_2$ is the Euclidean norm. We define the functions $f_1, f_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f_1(x,y) = (x, \eta_x y)$ and $f_2(x,y) = (x, \eta_x y + 1 - \eta_x)$ for all $(x,y) \in \mathbb{R}^2$, where

$$\eta_x = \begin{cases} \frac{1}{2}, \text{ if } x \leq 0 \\ \frac{1}{3}\cdot x + \frac{1}{2}\cdot(1-x), \text{ if } x \in (0,1) \\ \frac{1}{3}, \text{ if } x \geq 1. \end{cases}$$

We obtained that the system $\mathcal{S} = \left((\mathbb{R}^2, \|\cdot\|_2), (f_i)_{i\in\{1,2\}}\right)$ is an orbital contractive iterated function system. The set $A_{(0,0)}$ is connected and for $K = [0,1]\times\{0\} \in P_{cp}(\mathbb{R}^2)$, M_K and A_K are arcwise connected. The figures below represent the first steps in the construction of the attractor using the fractal operator and having as starting point the set $[0,1]^2$.

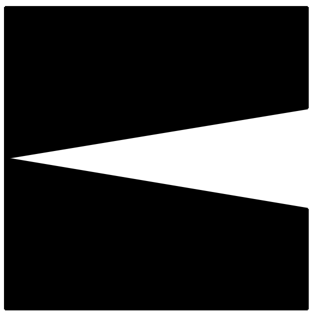


Example B. Let us consider the metric space $(X,d) = \left([0,1]^2, d_2\right)$, where d_2 is the

Euclidean distance. We define the functions f_0, f_1: $[0,1]^2 \to [0,1]^2$ given by

$$f_0(x,y) = \begin{cases} \left(x, \frac{y}{2}\right), \text{ if } y \le a_x \\ \left(x, \frac{a_x}{2}\right), \text{ if } y \in (a_x, 1-a_x) \\ \left(x, \frac{y-1+2a_x}{2}\right), \text{ if } y \ge 1-a_x \end{cases} \text{ and } f_1(x,y) = \begin{cases} \left(x, 1-\frac{y}{2}\right), \text{ if } y \le a_x \\ \left(x, \frac{2-a_x}{2}\right), \text{ if } y \in (a_x, 1-a_x) \\ \left(x, \frac{3-y-2a_x}{2}\right), \text{ if } y \ge 1-a_x \end{cases}$$

for all $x, y \in [0,1]$, where $a_x = \frac{1}{2}(1-x) + \frac{1}{3}x$ for every $x \in [0,1]$. We deduced that $\mathcal{S} = \left(\left([0,1]^2, \|\cdot\|_2\right), (f_i)_{i\in\{0,1\}}\right)$ is an orbital contractive iterated function system. For $K = [0,1]\times[0,1]$, A_K and M_K are arcwise connected. We have that $f_0(A_K)$ and $f_1(A_K)$ are connected. The attractor is represented in the following figure.



The above examples suggest the statement of Theorem 3.2, that gives us sufficient conditions for an attractor of an orbital contractive iterated function system to be arcwise connected.

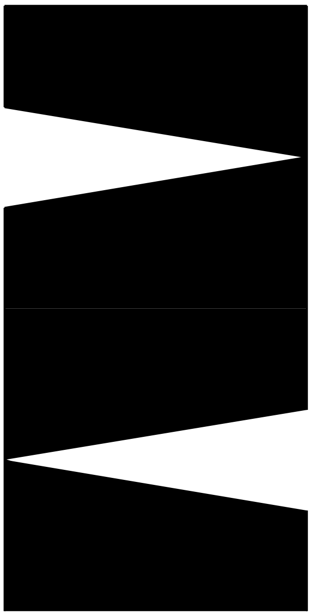
Theorem 3.2. *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be an orbital contractive iterated function system and* $K \in P_{cp}(X)$*, such that* M_K *is arcwise connected. If there exists* $x_0 \in A_K$ *such that* A_{x_0} *is connected, then* A_K *is arcwise connected.*

Theorem 3.3. *Let* $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ *be an orbital contractive iterated function system and* $K \in P_{cp}(X)$*. If* M_K *is arcwise connected and there exists* $x_0 \in K$ *such that* A_{x_0} *has a finite number of connected components, then* A_K *has a finite number of arcwise connected components, which is less than or equal to the number of connected components of* A_{x_0}*.*

In Theorem 3.3, the number of arcwise connected components of A_K is less than or equal to the number of connected components of A_{x_0}. In **Examples A** and **B**, A_K has an arcwise connected component and A_{x_0} has a connected component. In the following example (**Example C**) this equality doesn't take place (depending of $x \in K$, A_x has two or three connected components, while A_K has an arcwise connected component). In **Example C** we considered $K = [0,1] \times [0,2]$ and we obtained

$$A_K = \left\{ (x,y) \mid x \in [0,1],\ y \in \left[0, \frac{1}{2} - \frac{x}{6}\right] \cup \left[\frac{1}{2} + \frac{x}{6}, \frac{4}{3} + \frac{x}{6}\right] \cup \left[\frac{5}{3} - \frac{x}{6}, 2\right] \right\}.$$

The set A_K is arcwise connected and it is represented in the following figure:



Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an orbital contractive iterated function system and $K \in P_{cp}(X)$ and we suppose there exist $n \in \mathbb{N}^*$ and $x_1, \cdots, x_n \in K$ such that A_{x_j} has a finite number of connected components, denoted by $n_j \in \mathbb{N}^*$ for all $j \in \{1, \cdots, n\}$. The connected components of A_{x_j} are denoted by $A_{x_j}^1, \cdots, A_{x_j}^{n_j}$, for all $j \in \{1, \cdots, n\}$.

Let $\alpha, \beta \in \Lambda(I)$. We consider the relation $\perp$, defined by $\alpha \perp \beta$ if there exist $j \in \{1, \cdots, n\}$ and $k \in \{1, \cdots, n_j\}$ such that $a_\alpha(x_j)$, $a_\beta(x_j) \in A_{x_j}^k$. One can easily notice that, in general, $\perp$ is not an equivalence relation. We consider the equivalence relation $E_\perp = \bigcap_{E \text{ is an equivalence relation; } \perp \subset E} E$.

Theorem 3.4 (see [20]). *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an orbital contractive iterated function system and $K \in P_{cp}(X)$ such that M_K is arcwise connected. Assume that there exist $n \in \mathbb{N}^*$ and $x_1, \cdots, x_n \in K$ such that A_{x_j} has a finite number of connected components, for all $j \in \{1, \cdots, n\}$. If the relation $E_\perp$ has one class of equivalence, then A_K is arcwise connected.*

Chapter 4

φ-Contractive orbital affine iterated function systems

In this chapter we introduce the notion of φ-contractive orbital affine iterated function system (oAIFS). We present two results (Theorem 4.1 and Theorem 4.2) which give a structure result for the functions of an oAIFS and establish sufficient conditions to exist a norm with specific properties on the linear space where the functions are defined. We offer some examples. The results presented here were published in [22].

4.1 Preliminaries

If $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ are two normed spaces and $A \in L(Y, Z)$, by $\|A\|_{Y,Z}$ we mean $\sup\limits_{y \in Y, y \neq 0_Y} \frac{\|Ay\|_Z}{\|y\|_Y}$.

Let Y, $Z \subseteq \mathbb{R}^n$ be two real linear spaces such that $Y + Z = \mathbb{R}^n$ and $Y \cap Z = \{0_{\mathbb{R}^n}\}$. If $x \in \mathbb{R}^n$, then there exist a unique $y \in Y$ and a unique $z \in Z$ such that $y + z = x$. In this case, we shall use the notation $x = \begin{bmatrix} y \\ z \end{bmatrix}$. Let $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $x = \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^n$. We use the notation $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where $A_{11} \in L(Y, Y)$, $A_{12} \in L(Z, Y)$, $A_{21} \in L(Y, Z)$ and $A_{22} \in L(Z, Z)$ are defined by $Ay = \begin{bmatrix} A_{11}y \\ A_{21}y \end{bmatrix}$ and $Az = \begin{bmatrix} A_{12}z \\ A_{22}z \end{bmatrix}$ for every $y \in Y$ and $z \in Z$. Note that $Ax = \begin{bmatrix} A_{11}y \\ A_{21}y \end{bmatrix} + \begin{bmatrix} A_{12}z \\ A_{22}z \end{bmatrix}$ for every $\begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^n$.

Definition 4.1. On $\mathbb{R}^n$ we consider a fixed norm denoted by $\|\cdot\|$. By an affine iterated function system we mean a pair denoted by $\mathcal{S} = \left((\mathbb{R}^n, \|\cdot\|), (f_i)_{i\in I}\right)$, where $(f_i)_{i\in I}$ is a finite family of continuous functions, with $f_i\colon \mathbb{R}^n \to \mathbb{R}^n$ for all $i \in I$, having the property that for every $i \in I$, there exist $\widetilde{A}_i \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $\widetilde{a}_i \in \mathbb{R}^n$ such that $f_i(x) = \widetilde{A}_i x + \widetilde{a}_i$ for all $x \in \mathbb{R}^n$.

Definition 4.2. By a φ-contractive orbital affine iterated function system (oAIFS for short) we mean a pair denoted by $\mathcal{S} = \left((\mathbb{R}^n, \|\cdot\|), (f_i)_{i\in I}\right)$ which is an affine iterated function system and has the property that there exists a comparison function $\varphi\colon [0,\infty) \to [0,\infty)$ such that $\mathcal{S}$ is a φ-contractive orbital iterated function system.

4.2 Main results

Proposition 4.1 (see [22]). *Let $\mathcal{S} = \left((\mathbb{R}^n, \|\cdot\|), (f_i)_{i\in I}\right)$ be an oAIFS, $m \in \mathbb{N}$, $m \geq 2$ and $\alpha \in \Lambda_m(I)$. Then, $f_\alpha(x) = \widetilde{A}_\alpha x + \widetilde{a}_{\alpha_1} + \sum_{k=2}^{m} \widetilde{A}_{[\alpha]_{k-1}} \widetilde{a}_{\alpha_k}$ for all $x \in \mathbb{R}^n$.*

Theorem 4.1 (see [22]). *Let $\mathcal{S} = \left((\mathbb{R}^n, \|\cdot\|), (f_i)_{i\in I}\right)$ be an oAIFS. Then, there exist two linear subspaces $Y, Z \subset \mathbb{R}^n$ such that:*

1) $Y + Z = \mathbb{R}^n$, $Y \cap Z = \{0_{\mathbb{R}^n}\}$;

2) for every $i \in I$, there exist $B_i \in L(Z,Z)$, $C_i \in L(Y,Z)$ and $b_i \in Z$ such that $\widetilde{A}_i = \begin{bmatrix} I_Y & O_{Z,Y} \\ C_i & B_i \end{bmatrix}$ and $\widetilde{a}_i = \begin{bmatrix} 0_Y \\ b_i \end{bmatrix}$;

3) there exist $c \in (0,1)$ and a norm $\|\cdot\|_Z$ on Z such that $\|B_i\|_Z < c$ for all $i \in I$.

Theorem 4.2 (see [22]). *Let $\mathcal{S} = \left((\mathbb{R}^n, \|\cdot\|), (f_i)_{i\in I}\right)$ be an oAIFS. Let Y and Z be the linear subspaces of $\mathbb{R}^n$ which result from Theorem 4.1 and let $\|\cdot\|_Y$ be a norm defined on Y. Let $\mu = \max_{i\in I} \|B_i\|_Z$, $\beta = \max_{i\in I} \|C_i\|_{Y,Z}$ and $\theta \in \left(0, \frac{1-\mu}{\beta}\right)$. We consider the norm $\|\cdot\|_\theta \colon \mathbb{R}^n \to [0,\infty)$ defined by*

$$\left\| \begin{bmatrix} y \\ z \end{bmatrix} \right\|_\theta = \max\{\|y\|_Y, \theta \|z\|_Z\}$$

for all $y \in Y$ and $z \in Z$ and the norm $|||\cdot|||\colon Z \to [0,\infty)$ given by $|||z||| = \theta||z||_Z$ for all $z \in Z$. Then, $\left\|\widetilde{A}_i\right\|_\theta \leq 1$ and $|||B_i||| = \|B_i\|_Z < 1$ for all $i \in I$.

Chapter 5

Orbital fuzzy iterated function systems

In this chapter we introduce the notion of orbital fuzzy iterated function system. We prove that the fuzzy Hutchinson-Barnsley operator associated with such a system is weakly Picard. Also, for each fuzzy set, we provide a description of the corresponding fuzzy fractal. In addition, we study the fuzzy fractal generated by a canonical fuzzy iterated function system and give a structure result regarding the fuzzy fractals generated by an orbital fuzzy iterated function system. Some examples are provided. The results presented here can be found in [15], [16] and [21].

5.1 Preliminaries

The terminology used in this chapter can be found in [23].

Let (X, d) be a complete metric space. The family of fuzzy subsets of X is denoted by $\mathcal{F}_X$. For $u \in \mathcal{F}_X$, we use the following notation: $[u]^*:=\{x \in X \mid u(x) > 0\}$. An element $u \in \mathcal{F}_X$ is called normal if there exists $x \in X$ such that $u(x) = 1$.

Notation 5.1. $\mathcal{F}_X^{**} = \{u \in \mathcal{F}_X \mid u \text{ is normal and compactly supported}\}$;
$\mathcal{F}_X^{*} = \{u \in \mathcal{F}_X^{**} \mid u \text{ is usc (upper semicontinuous)}\}$.

The topology on $\mathcal{F}_X^{**}$ is defined using the Hausdorff-Pompeiu semidistance between the α-cuts. Since $P_b(X)$ contains all the α-cuts, we can define a semimetric d_∞ in $\mathcal{F}_X^{**}$ by $d_\infty(u, v) = \sup\limits_{\alpha \in [0,1]} h([u]^\alpha, [v]^\alpha)$ for every $u, v \in \mathcal{F}_X^{**}$. The restriction of d_∞ to $\mathcal{F}_X^{*}$ is a metric, since in this case the α-cuts belong to $P_{cp}(X)$.

Given a metric space (X, d) and $(f_i)_{i \in I}$ a family of functions with $f_i\colon X \to \mathbb{R}$ for all $i \in I$, we denote by $\bigvee\limits_{i \in I} f_i\colon X \to \mathbb{R}$ the function given by $(\bigvee\limits_{i \in I} f_i)(x) = \sup\limits_{i \in I} f_i(x)$ for every $x \in X$.

Definition 5.1. Let $\mathcal{S} = ((X,d),(f_i)_{i\in I})$ be a contractive iterated function system and let $(\rho_i)_{i\in I}$ be an admissible system of grey level maps. The system denoted by $\mathcal{S}_Z = \left((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I}\right)$ is called an iterated fuzzy function system.

Definition 5.2. Let $\mathcal{S} = \left((X,d),(f_i)_{i\in I}\right)$ be an orbital contractive iterated function system and let $(\rho_i)_{i\in I}$ be an admissible system of grey level maps. The system $\mathcal{S}_Z = \left((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I}\right)$ is called an orbital fuzzy iterated function system.

If $\mathcal{S}_Z = \left((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I}\right)$ is an iterated fuzzy function system or an orbital fuzzy iterated function system, the function $Z\colon \mathcal{F}_X^{**} \to \mathcal{F}_X^{**}$, given by $Z(u) = \underset{i\in I}{\vee}\rho_i(f_i(u))$ for all $u \in \mathcal{F}_X^{**}$, is called the fuzzy Hutchinson Barnsley operator associated with $\mathcal{S}_Z$.

The function $\hat{Z}\colon \mathcal{F}_X^{*} \to \mathcal{F}_X^{*}$, is given by $\hat{Z}(u) = Z(u)$ for all $u \in \mathcal{F}_X^{*}$. For the sake of simplicity, we will denote $\hat{Z}$ by Z.

5.2 Results regarding the fuzzy operator associated with an orbital fuzzy iterated function system

Let $\mathcal{S}_Z = \left((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I}\right)$ be an orbital fuzzy iterated function system. We consider $\mathcal{F}_{\mathcal{S}}^{**} = \{u \in \mathcal{F}_X^{**}|$ if for those $x \in X$ with the property $u(x) > 0$, there exist $w \in X$ and $y \in \overline{\mathcal{O}(w)}$ such that $x \in \overline{\mathcal{O}(w)}$ and $u(y) = 1\}$ and $\mathcal{F}_{\mathcal{S}}^{*} = \{u \in \mathcal{F}_{\mathcal{S}}^{**}|\ u \text{ is usc}\}$.

We note that the operator $Z\colon \mathcal{F}_X^{**} \to \mathcal{F}_X^{**}$ induces the operator $\mathbf{Z}\colon \mathcal{F}_{\mathcal{S}}^{**} \to \mathcal{F}_{\mathcal{S}}^{**}$ given by $\mathbf{Z}(u) = Z(u)$ for every $u \in \mathcal{F}_{\mathcal{S}}^{**}$, which, for the sake of simplicity, will be denoted by Z instead of $\mathbf{Z}$.

Also, $Z\colon \mathcal{F}_X^{*} \to \mathcal{F}_X^{*}$ induces the operator $\hat{\mathbf{Z}}\colon \mathcal{F}_{\mathcal{S}}^{*} \to \mathcal{F}_{\mathcal{S}}^{*}$ given by $\hat{\mathbf{Z}}(u) = Z(u)$ for every $u \in \mathcal{F}_{\mathcal{S}}^{*}$, which, for the sake of simplicity, will be denoted by Z instead of $\hat{\mathbf{Z}}$.

Lemma 5.1 (see [21]). *Let $\mathcal{S}_Z = \left((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I}\right)$ be an orbital fuzzy iterated function system, with I an nonempty finite set. Then $(\mathcal{F}_{\mathcal{S}}^{*}, d_\infty)$ is a closed subset of $(\mathcal{F}_X^{*}, d_\infty)$.*

Theorem 5.1 (see [21]). *The fuzzy operator associated with an orbital fuzzy iterated function system $\mathcal{S}$ is a weakly Picard operator on $\mathcal{F}_{\mathcal{S}}^{*}$.*

Example A. Let us consider the metric space $(\mathbb{R}^2, \|\cdot\|_2)$, where $\|\cdot\|_2$ is the Euclidean norm, and the functions f_1 and f_2, with $f_1,\ f_2\colon \mathbb{R}^2 \to \mathbb{R}^2$, given by $f_1(x,y) = \left(x, \frac{1}{2}y\right)$ and $f_2(x,y) = \left(x, \frac{1}{2}y + \frac{1}{2}\right)$ for all $(x,y) \in \mathbb{R}^2$. Let us consider $I = \{1,2\}$. One can easily prove that $\mathcal{S} = \left((\mathbb{R}^2, \|\cdot\|_2),(f_i)_{i\in I}\right)$ is an orbital iterated function system and for every $K \in P_{cp}(\mathbb{R}^2)$, the attractor is $A_K = [0,1] \times \pi_2(K)$ (see **example A** from

[25]). We define an admissible system of functions $(\rho_i)_{i\in I}$, given by $\rho_1(t) = t$ and $\rho_2(t) = \begin{cases} 0, \text{ if } t\in \left[0,\frac{1}{3}\right) \\ \frac{1}{3}, \text{ if } t\in \left[\frac{1}{3},\frac{2}{3}\right) \\ \frac{2}{3}, \text{ if } t\in \left[\frac{2}{3},1\right] \end{cases}$ for every $t\in[0,1]$. We consider the function $u\in\mathcal{F}^*_{\mathcal{S}_\Lambda}$, defined by $u(x,y) = \begin{cases} 1, \text{ if } (x,y)\in[0,1]\times\{0\} \\ 0, \text{ otherwise} \end{cases}$, for all $(x,y)\in\mathbb{R}^2$.

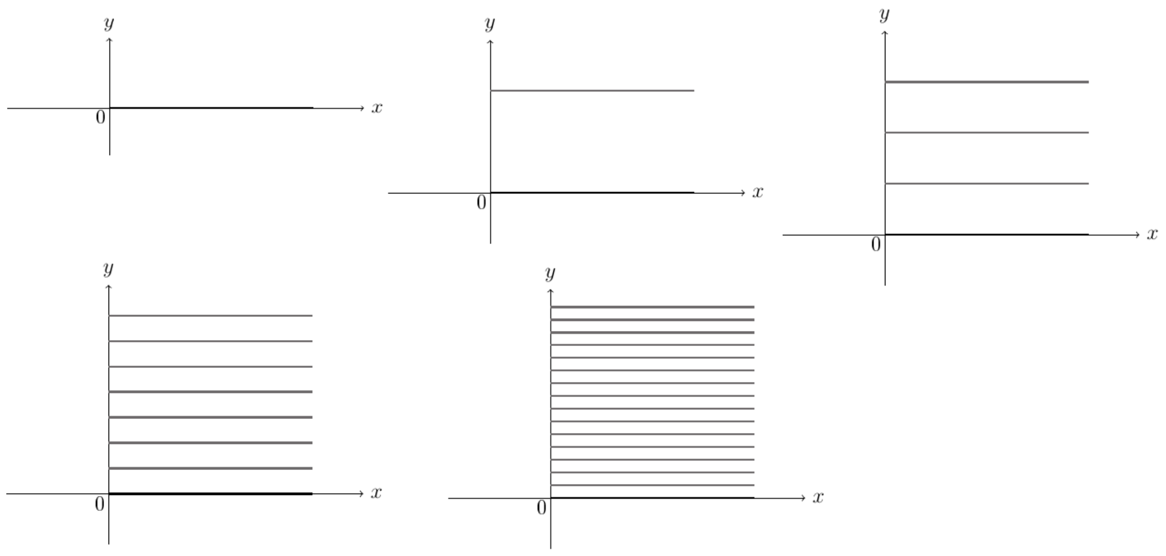
Let $x\in[0,1]$. We now consider the system $\mathcal{S}$ restricted to $X_x\colon = \{x\}\times\mathbb{R}$. One can easily prove that it is a contractive iterated function system. In this case, the attractor is $A_x = \{x\}\times[0,1]$. We are going to give details on $\mathbf{u}_u$.

Using the mathematical induction method, one can prove that

$$Z^n(u)(x,y) = \begin{cases} 1, \text{ if } x\in[0,1] \text{ and } y=0 \\ \frac{2}{3}, \text{ if } x\in[0,1] \text{ and } y\in\left\{\frac{p}{2^n} \mid p\in\{0,1,2,\cdots,2^n-1\}\right\} \\ 0, \text{ otherwise} \end{cases},$$

for all $(x,y)\in\mathbb{R}^2$ and $n\in\mathbb{N}^*$. By passing to limit, we have $\mathbf{u}_u = \lim\limits_{n\to\infty} Z^n(u)$.

The following figures illustrate the above example. The first step represents u, the second step illustrates $Z(u)$, in the third step is represented $Z^2(u)$, the fourth by $Z^3(u)$ and in the last step we illustrated $Z^4(u)$.



Example B. We consider the system $\mathcal{S} = \left((\mathbb{R}^2, \|\cdot\|_2), (f_i)_{i\in I}\right)$ presented in **Example A**. Let us define the admissible system of functions $(\rho_i)_{i\in I}$, given by $\rho_1(t) = t$ and

$\rho_2(t) = \frac{3t}{4}$ for every $t \in [0,1]$.

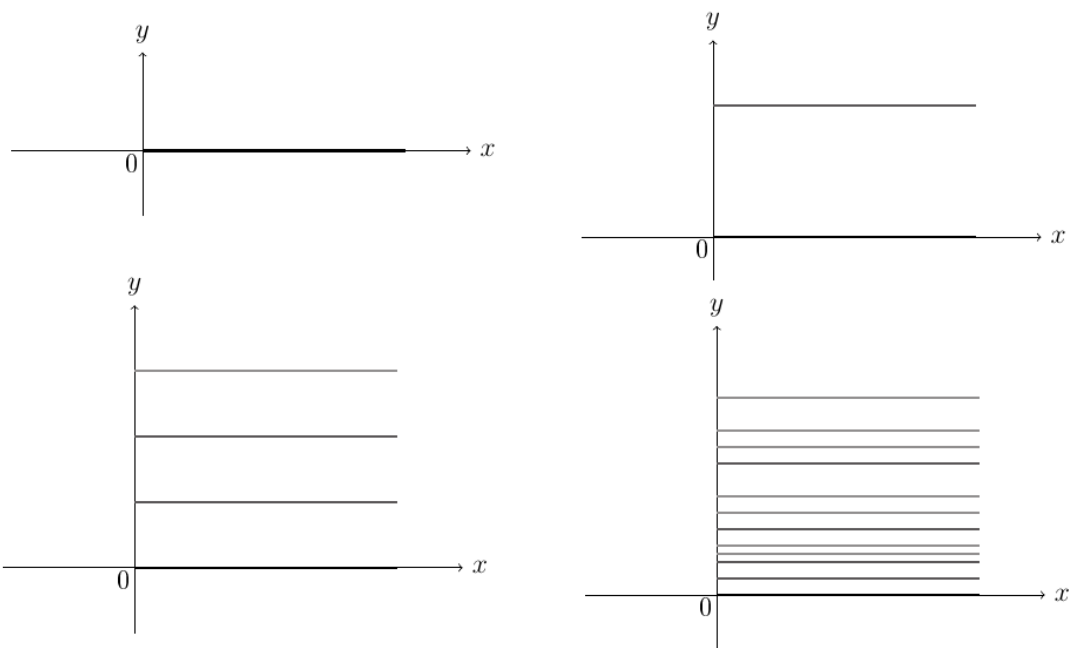
Let $x \in [0,1]$. We now consider the system $\mathcal{S}$ restricted to $X_x \colon = \{x\} \times \mathbb{R}$. We consider the function $u \in \mathcal{F}^*_{\mathcal{S}_\Lambda}$ from **Example A** and we want to describe the function $\mathbf{u}_u$. Let $p \colon \Lambda^*(I) \cup \Lambda(I) \to [0,1]$ be given by $p(\alpha) = \sum_{n=1}^{|\alpha|} \frac{\alpha_n - 1}{2^n}$, for all $\alpha = \alpha_1\alpha_2 \cdots \alpha_n \cdots \in \Lambda^*(I) \cup \Lambda(I)$. We also define the function $\eta \colon \Lambda^*(I) \cup \Lambda(I) \to \mathbb{N} \cup \{+\infty\}$, given by $\eta(\alpha) = |\{i \mid \alpha_i = 2,\ 1 \le i \le |\alpha|\}|$ for all $\alpha = \alpha_1\alpha_2 \cdots \in \Lambda^*(I) \cup \Lambda(I)$, where for a set A we denoted by $|A|$ the number of elements of A. Using the mathematical induction method, one can prove that

$$Z^n(u)(x,y) = \max_{p(\alpha)=y, |\alpha| \le n} \left(\frac{3}{4}\right)^{\eta(\alpha)},$$

if $(x,y) \in [0,1]^2$ and there exists $\alpha \in \Lambda^*(I)$ such that $|\alpha| \le n$, $y = p(\alpha)$ and $Z^n(u)(x,y) = 0$, otherwise.

By passing to limit, we have $\mathbf{u}_u(x,y) = \lim_{n\to\infty} Z^n(u)(x,y) = \max_{p(\alpha)=y, \alpha \in \Lambda^*(I)\cup\Lambda(I)} \left(\frac{3}{4}\right)^{\eta(\alpha)}$ for every $(x,y) \in \mathbb{R}^2$.

The following figures illustrate the example presented above. The first step represents u, the second step illustrates $Z(u)$, in the third step it is represented $Z^2(u)$ and in the last step we illustrated a part of $Z^4(u)$.



5.3 A characterization of the fuzzy fractals generated by an orbital fuzzy iterated function system

Let us fix $u \in \mathcal{F}_{\mathcal{S}}^*$ and $x \in [u]^*$. Hence, there exist $w_x, y_x \in X$ such that $x, y_x \in \overline{\mathcal{O}(w_x)}$ and $u(y_x) = 1$. Let us fix w_x as above. We consider the function $u^x\colon X \to [0,1]$ defined as in Theorem 5.1, namely $u^x(y) = \begin{cases} u(y), \text{ if } y \in \overline{\mathcal{O}(w_x)} \\ 0, \text{ otherwise} \end{cases}$, for every $y \in X$. We have $u^x \in \mathcal{F}_{\mathcal{S}}^*$.

We make the notations: $\mathbf{u}_u \colon = \lim\limits_{n\to\infty} Z^n(u)$, $w_u \colon = \bigvee\limits_{x\in[u]^*} \mathbf{u}_{x,w_x,u}$ and $\mathbf{u}_{x,w_x,u} = \lim\limits_{n\to\infty} Z^n(u^x)$.

The existence of the above limits is based on Theorem 5.1.

For $x \in X$, we consider $\delta_x \in \mathcal{F}_X^*$ given by $\delta_x(t) = \begin{cases} 1, \text{ if } t = x \\ 0, \text{ if } t \neq x \end{cases}$, for every $t \in X$.

Note that $\delta_x \in \mathcal{F}_{\mathcal{S}}^*$ for every $x \in X$.

Proposition 5.1 (see [15]). *Let $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system, $u \in \mathcal{F}_{\mathcal{S}}^*$ and $x \in [u]^*$ (hence there exist $w_x, y_x \in X$ such that $x, y_x \in \overline{\mathcal{O}(w_x)}$ and $u(y_x) = 1$). Then $\lim\limits_{n\to\infty} Z^n(\delta_s) = \mathbf{u}_{x,w_x,u}$, for every $s \in \overline{\mathcal{O}(w_x)}$. In particular, $\lim\limits_{n\to\infty} Z^n(\delta_x) = \mathbf{u}_{x,w_x,u}$.*

Proposition 5.1 ensures us that $\mathbf{u}_{x,w_x,u}$ does not depend on the function u. Hence, instead of $\mathbf{u}_{x,w_x,u}$, we will use the notation $\mathbf{u}_x$.

Proposition 5.2 (see [15]). *Let $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system and $u \in \mathcal{F}_{\mathcal{S}}^*$. Then $\mathbf{u}_y = \mathbf{u}_x$, for every $x \in [u]^*$ and every $y \in [\mathbf{u}_x]^*$.*

Proposition 5.3 (see [15]). *Let $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system and $u \in \mathcal{F}_{\mathcal{S}}^*$. Then the function $U\colon [u]^* \to \mathcal{F}_X^*$, given by $U(x) = \mathbf{u}_x$, for every $x \in [u]^*$, is continuous.*

Proposition 5.4 (see [15]). *Let $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system and $u \in \mathcal{F}_{\mathcal{S}}^*$. Then $[\bigvee\limits_{x\in[u]^*} \mathbf{u}_x]^\alpha = \bigcup\limits_{x\in[u]^*} [\mathbf{u}_x]^\alpha$, for every $\alpha \in (0,1]$ and $u \in \mathcal{F}_{\mathcal{S}}^*$.*

Proposition 5.5 (see [15]). *Let $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system and $u \in \mathcal{F}_{\mathcal{S}}^*$. Then $\{\mathbf{u}_x \mid x \in [u]^*\} = \{\mathbf{u}_x \mid x \in [u]^1\}$. In particular, $w_u = \bigvee\limits_{x\in[u]^*} \mathbf{u}_x = \bigvee\limits_{x\in[u]^1} \mathbf{u}_x$.*

Proposition 5.6 (see [15]). *Let $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system and $u \in \mathcal{F}_{\mathcal{S}}^*$. Then w_u is upper semicontinuous, so $w_u \in \mathcal{F}_X^*$.*

Proposition 5.7 (see [15]). *Let* $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ *be an orbital fuzzy iterated function system and* $u \in \mathcal{F}^*_{\mathcal{S}}$. *Then* $d_\infty(\mathbf{u}_u, w_u) = 0$.

Theorem 5.2 (see [15]). *Let* $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ *be an orbital fuzzy iterated function system and* $u \in \mathcal{F}^*_{\mathcal{S}}$. *Then* $\mathbf{u}_u = \bigvee_{x\in[u]^*} \mathbf{u}_x = \bigvee_{x\in[u]^1} \mathbf{u}_x = \max_{x\in[u]^*} \mathbf{u}_x = \max_{x\in[u]^1} \mathbf{u}_x$.

5.4 The structure of fuzzy fractals generated by an orbital fuzzy iterated function system

Let $\mathcal{S}_Z = \left((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I}\right)$ be an iterated fuzzy function system and let Z be the fuzzy Hutchinson-Barnsley operator associated with $\mathcal{S}_Z$. We denote by $\mathbf{u}_{\mathcal{S}}$ the fuzzy fractal generated by $\mathcal{S}_Z$.

We consider the canonical iterated fuzzy function system which is denoted by $\mathcal{S}_\Lambda = ((\Lambda(I),d_c),(\tau_i)_{i\in I},(\rho_i)_{i\in I})$. The fuzzy Hutchinson-Barnsley operator associated with $\mathcal{S}_\Lambda$ will be denoted by Z_Λ and the fuzzy fractal of Z_Λ will be denoted by $\mathbf{u}_\Lambda$. Hence, $Z_\Lambda(\mathbf{u}_\Lambda) = \mathbf{u}_\Lambda$ and $\lim_{n\to\infty} d_\infty(Z^n_\Lambda(u), \mathbf{u}_\Lambda) = 0$, for every $u \in \mathcal{F}^*_{\Lambda(I)}$.

Theorem 5.3 (see [16]). *In the above framework, we have* $\mathbf{u}_\Lambda = \lim_{n\to\infty} u_n$, *where* $u_n \in \mathcal{F}^*_{\mathcal{S}_\Lambda}$ *is given by* $u_n(\omega) = \rho_{[\omega]_n}(1)$, *for every* $n \in \mathbb{N}^*$ *and every* $\omega \in \Lambda(I)$.

Theorem 5.4 (see [16]). *In the above framework, for every iterated fuzzy function system* $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$, *we have* $\mathbf{u}_S = \pi(\mathbf{u}_\Lambda)$, *where* π *is the canonical projection associated with the contractive iterated function system* $((X,d),(f_i)_{i\in I})$.

Let $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system and let $u \in \mathcal{F}^*_{\mathcal{S}}$. Then, for every $x \in [u]^*$, there exist $w_x, y_x \in X$ such that $x, y_x \in \overline{\mathcal{O}(w_x)}$ and $u(y_x) = 1$.

We consider the iterated fuzzy function system $((\overline{\mathcal{O}(w_x)},d),(\widetilde{f_i})_{i\in I},(\rho_i)_{i\in I}) \overset{not}{=} S_{w_x}$, where $\widetilde{f_i}\colon \overline{\mathcal{O}(w_x)} \to \overline{\mathcal{O}(w_x)}$ is given by $\widetilde{f_i}(y) = f_i(y)$, for every $y \in \overline{\mathcal{O}(w_x)}$. We denote by π_x its canonical projection and by Z_{w_x} its fuzzy Hutchinson-Barnsley operator. Let us also denote by $\widetilde{\pi_x}(\mathbf{u}_\Lambda)$ the function given by $\widetilde{\pi_x}(\mathbf{u}_\Lambda)(y) = \begin{cases} \pi_x(\mathbf{u}_\Lambda)(y), \text{ if } y \in \overline{\mathcal{O}(w_x)} \\ 0, \text{ if } y \in X \setminus \overline{\mathcal{O}(w_x)} \end{cases}$ for every $y \in X$.

Theorem 5.5 (see [16]). *In the above framework, we have* $\mathbf{u}_u = \bigvee_{x\in[u]^*} \widetilde{\pi_x}(\mathbf{u}_\Lambda) = \bigvee_{x\in[u]^1} \widetilde{\pi_x}(\mathbf{u}_\Lambda) = \max_{x\in[u]^*} \widetilde{\pi_x}(\mathbf{u}_\Lambda) = \max_{x\in[u]^1} \widetilde{\pi_x}(\mathbf{u}_\Lambda)$.

References

1. M. F. Barnsley, *Fractals everywhere*, Academic Press, Boston, MA, 1988.

2. M. Boriceanu, M. Bota, A Petruşel, *Multivalued fractals in b-metric spaces*, Cent. Eur. J. Math., **8** (2010), no. 2, 367-377.

3. D. Dumitru, L. Ioana, R. C. Sfetcu, F. Strobin, *Topological version of generalized (infinite) iterated function systems,* Chaos Solitons Fractals, **71** (2015), 78-90.

4. N. Van Dung, A. Petruşel, *On iterated function systems consisting of Kannan maps, Reich maps, Chatterjea type maps, and related results*, J. Fixed Point Theory Appl., **19** (2017), no. 4, 2271-2285.

5. H. Fernau, *Infinite iterated function systems*, Math. Nachr., **170** (1994), 79-91.

6. G. Gwóźdź-Lukowska, J. Jachymski, *The Hutchinson-Barnsley theory for infinite iterated function systems*, Bull. Austral. Math. Soc., **72** (2005), no. 3, 441-454.

7. J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J., **30** (1981), no. 5, 713-747.

8. I. Jaksztas, *Infinite iterated function systems depending on a parameter*, Bull. Pol. Acad. Sci. Math., **55** (2007), no. 2, 105-122.

9. S. Kumari, K. Gdawiec, A. Nandal, M. Postolache, R. Chugh, *A novel approach to generate Mandelbrot sets, Julia sets and biomorphs via viscosity approximation method,* Chaos Solitons Fractals **163** (2022), no. 112540, 21 pp.

10. H. Kunze, D. La Torre, E. R. Vrscay, *From iterated function systems to iterated multifunction systems*, Comm. Appl. Nonlinear Anal. **15** (2008), no. 4, 1-13.

11. A. Lasota, J. Myjak, *Attractors of multifunctions*, Bull. Polish Acad. Sci. Math., **48** (2000), no. 3, 319-334.

12. K. Leśniak, N. Snigireva, F. Strobin, *Weakly contractive iterated function systems and beyond: a manual*, J. Difference Equ. Appl., **26** (2020), no. 8, 1114-1173.

13. F. Mendivil, *A generalization of IFS with probabilities to infinitely many maps*, Rocky Mountain J. Math., **28** (1998), no. 3, 1043-1051.

14. R. Miculescu, A. Mihail, **I. Savu**, *Iterated function systems consisting of continuous functions satisfying Banach's orbital condition,* An. Univ. Vest Timiş. Ser. Mat.-Inform., **56** (2018), no. 2, 71-80.

15. R. Miculescu, A. Mihail, **I. Savu**, *A characterization of the fuzzy fractals generated by an orbital fuzzy iterated function system*, Carpathian J. Math., **38** (2022), no. 3, 583-595.

16. R. Miculescu, A. Mihail, **I. Savu**, *The structure of fuzzy fractals generated by an orbital fuzzy iterated function system*, submitted.

17. A. Mihail, *The canonical projection between the shift space of an IIFS and its attractor as a fixed point,* Fixed Point Theory Appl., **75** (2015), 15 pp.

18. A. Mihail, **I. Savu**, *φ-Contractive parent-child possibly infinite IFSs and orbital φ-contractive possibly infinite IFSs*, accepted for publication, will appear in Fixed Point Theory; arXiv:2103.07551.

19. A. Mihail, **I. Savu**, *Orbital φ-contractive iterated function systems*, Proceedings of Research World International Conference, Prague, Czech Republic, 21^{st}-22^{nd} September 2020.

20. A. Mihail, **I. Savu**, *On the connectedness of attractors of orbital contractive IFSs*, Topology Appl., **326** (2023), Paper No. 108412, 16 pp.

21. A. Mihail, **I. Savu**, *Orbital fuzzy iterated function systems,* Fuzzy Sets and Systems, https://doi.org/10.1016/j.fss.2023.02.012; arXiv:2112.15496.

22. A. Mihail, **I. Savu**, *φ-Contractive orbital affine iterated function systems*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **85** (2023), no. 1, 13-24.

23. E. R. Oliveira, F. Strobin, *Fuzzy attractors appearing from GIFZS,* Fuzzy Sets and Systems, **331** (2018), 131-156.

24. A. Petruşel, A. Soos, *Self-similar sets and fractals generated by Ćirić type operators*, J. Nonlinear Sci. Appl., **8** (2015), no. 6, 1048-1058.

25. **I. Savu**, *New aspects concerning IFSs consisting of continuous functions satisfying Banach's orbital condition*, J. Fixed Point Theory Appl., **21** (2019), no. 2, Paper No. 62, 11 pp.

26. N. A. Secelean, *Iterated function systems consisting of F-contractions,* Fixed Point Theory Appl., **277** (2013), 13 pp.

27. N. A. Secelean, *Countable iterated function systems*, Lambert Academic Publishing, 2013.

28. N. A. Secelean, *Generalized iterated function systems on the space* $l^{\infty}(X)$, J. Math. Anal. Appl., **410** (2014), no. 2, 847-858.

29. N. A. Secelean, *Invariant Measure Associated with a Generalized Countable Iterated Function System*, Mediterr. J. Math., **11** (2014), no. 2, 361-372.

30. D. La Torre, F. Mendivil, E. R. Vrscay, *Iterated Function Systems on Multifunctions*, Math everywhere, 125-138, Springer, Berlin, 2007.

31. S.A. Urziceanu, *Possibly infinite generalized iterated function systems comprising* φ*-max contractions*, Stud. Univ. Babeş-Bolyai Math., **64** (2019), no. 2, 139-150.

32. R. Zaharopol, *Equicontinuity and existence of attractive probability measures for some iterated function systems*, Rev. Roumaine Math. Pures Appl., **52** (2007), no. 2, 259-286.