UNIVERSITY **POLITEHNICA** OF BUCHAREST
DOCTORAL SCHOOL OF APPLIED SCIENCES

PH-D THESIS SUMMARY

On the categories of fuzzy coverings and tolerance relations

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Keywords: fuzzy set, fuzzy covering, normal fuzzy covering, non-inclusive fuzzy covering, disjoint fuzzy covering, fuzzy relation, fuzzy tolerance relation, fuzzy partition, category, isomorphism

Bucharest
2023

Contents

Summary

Fuzzy sets were first introduced by Lotfi Zadeh [26] in 1965. Fuzzy sets are generalizations of sets. The statement that an element is in a fuzzy set can be neither false nor true, but something in between. For a comprehensive introduction we refer the reader to [17] and [18].

Definition 2.1.1. *(See also [7, Definition 1])*

Let X be a set. We say that $(X, (A_i)_{i\in I})$ is a fuzzy covering*, or a* covering *of X, if $A_i : X \to [0, 1]$ are fuzzy sets such that for all $x \in X$, there exists $i \in I$ with $A_i(x) = 1$.*

The notion of a covering is similar to a set of attributes such that every element of the set has at least one attribute. Coverings have a high importance in fuzzy control and machine learning. There is a vast literature on this topic: See [21], and [8], just te mention a few.

Definition 2.1.17. *(See also [1, Definition 2])*

Let I be a finite set of indices. We say that $(X, (B_i)_{i\in I})$ is a fuzzy partition*, or, simply, a* partition *of X, if $B_i : X \to [0, 1]$ are fuzzy sets such that $\sum_{i\in I} B_i(x) = 1$ for all $x \in X$.*

Note that a fuzzy partition resemble a stochastic process with a finite state space.

A (fuzzy) relation R on a set X is a $X \times X$ (fuzzy) subset ([19], [24], [12]). If the (fuzzy) relation is reflexive and symmetric then it is called a (fuzzy) tolerance relation. The tolerance relations were first studied by Zeeman [27] and have been pursued in [3], [4], [6], [9], [22] and [23].

The purpose of this thesis is to use a category theory approach ([2], [11], [25]) in the study of fuzzy coverings, fuzzy tolerance relations and fuzzy partitions ([3], [7]) and to establish relations between them.

The thesis is structured as follows. In the first chapter, we recall basic definitions and fact regarding fuzzy sets, fuzzy relations, categories and functors. In the first section of the second chapter, we present an introduction in the theory of fuzzy coverings and partitions. In Section 2.2 we introduce the notions of permutation of coverings and inclusion of coverings and prove several results about them.

Definition 2.2.1. *We say that the covering* $\left(X, (B_j)_{j\in J}\right)$ *is a* permutation *of the covering* $\left(X, (A_i)_{i\in I}\right)$ *and we write* $\left(X, (A_i)_{i\in I}\right) \simeq \left(X, (B_j)_{j\in J}\right)$ *if there exists a bijective function* $\rho : I \to J$ *such that*

$$A_i(x) = B_{\rho(i)}(x) \text{ for all } x \in X \text{ and } i \in I.$$

We prove that $\simeq$ is a relation of equivalence; see Proposition 2.2.2.

Definition 2.2.4. *Let* $\left(X, (A_i)_{i\in I}\right)$ *and* $\left(X, (B_j)_{j\in J}\right)$ *be two coverings. We say that the covering* $\left(X, (A_i)_{i\in I}\right)$ *is* included *in the covering* $\left(X, (B_j)_{j\in J}\right)$ *and we denote it by*

$$\left(X, (A_i)_{i\in I}\right) \subseteq \left(X, (B_j)_{j\in J}\right),$$

if there exists a function $\rho : I \to J$ *such that for all* $x \in X$ *and* $i \in I$ *we have:*

$$A_i(x) \leq B_{\rho(i)}(x).$$

ρ *is called the* inclusion function *associated to the covering inclusion.*

In Proposition 2.2.6, we prove that the relation of inclusion is reflexive and transitive.

In Section 2.3, we introduce and study new types of fuzzy coverings: normal coverings, non-inclusive coverings and disjoint coverings.

Definition 2.3.1. *A* normal *covering* $\left(X, (A_i)_{i\in I}\right)$ *is a covering with the property that for all* $i \in I$, *there exists* $x \in X$ *such that* $A_i(x) = 1$.

Definition 2.3.6. *We say that* $\left(X, (A_i)_{i\in I}\right)$ *is a* non-inclusive covering *if for any* $i, j \in I$:

$$A_i(x) \leq A_j(x), \text{for all } x \in X \text{ if and only if } i = j.$$

A covering which is not non-inclusive it is called inclusive.

We note that a partition is a non-inclusive covering. In Theorem 2.3.9 we prove that the relation of inclusion of coverings is an order relation on non-inclusive coverings, modulo the relation of equivalence of coverings.

Definition 2.3.10. *We say that* $\left(X, (A_i)_{i\in I}\right)$ *is a* disjoint covering *if for any* $x \in X$ *and* $i \neq j \in I$ *we have:*

$$A_i(x) \wedge A_j(x) < 1.$$

Theorem 2.3.14. *Let* $\left(X, (A_i)_{i\in I}\right)$ *be a covering. The following are equivalent:*

(1) $\left(X, (A_i)_{i\in I}\right)$ *is a disjoint normal covering.*

(2) $\left(X, \left(A_i^{\downarrow}\right)_{i\in I}\right)$ *is a partition.*

In Section 2.4 we introduce projection coverings and we establish basic properties of them.

Definition 2.4.1. *A* projection covering *of* $X \times Y$ *is a pair* $\left(X \times Y, (A_i)_{i\in I}\right)$ *such that* $\Big(X, (A_i(-, y))_{(y,i)\in Y\times I}\Big)$ *and* $\Big(Y, (A_i(x, -))_{(x,i)\in X\times I}\Big)$ *are coverings.*

Given a projection covering $\left(X \times Y, (A_i)_{i\in I}\right)$, we can associate a pair of functions: $f : X \to Y$ and $g : Y \to X$, such that, for all $x \in X$ there exists $i \in I$ with $A_i(x, f(x)) = 1$ and for all $y \in Y$ there exists $j \in J$ such that $B_j(g(y), y) = 1$. f and g are called *subfunctions*. In general, they are not unique.

However, in Theorem 2.4.12. we prove that if f and g are unique, then f and g are bijective and $f^{-1} = g$.

In the third chapter, we introduce and study several categories of fuzzy coverings, fuzzy relations and fuzzy partitions. In Section 3.1 we introduce **Covering**, the category of fuzzy coverings, as follows:

Definition 3.1.1. *Let* **Covering** *be the category which has:*

(1) $\mathrm{Ob}(\textbf{Covering}) = \left\{\left(X, (A_i)_{i\in I}\right) \mid \left(X, (A_i)_{i\in I}\right) \text{ is a fuzzy covering}\right\}$.

(2) $\mathrm{Hom}\left(\left(X, (A_i)_{i\in I}\right), \left(Y, (B_j)_{j\in J}\right)\right) = \{(f, \rho) | f : X \to Y, \rho : I \to J,$
$A_i(x) \le B_{\rho(i)}(f(x)),$ *for all* $x \in X, i \in I\}$.

(3) $(g, \theta) \circ (f, \rho) = (g \circ f, \theta \circ \rho) \in \mathrm{Hom}\left(\left(X, (A_i)_{i\in I}\right), \left(Z, (C_k)_{k\in K}\right)\right)$
for all $(f, \rho) \in \mathrm{Hom}\left(\left(X, (A_i)_{i\in I}\right), \left(Y, (B_j)_{j\in J}\right)\right)$,
$(g, \theta) \in \mathrm{Hom}\left(\left(Y, (B_j)_{j\in J}\right), \left(Z, (C_k)_{k\in K}\right)\right)$.

(4) $\mathrm{id}_{\left(X,(A_i)_{i\in I}\right)} = (\mathrm{id}_X, \mathrm{id}_I)$, *for all* $\left(X, (A_i)_{i\in I}\right) \in \mathrm{Ob}(\textbf{Covering})$.

The main result of the section is a theorem where we determine the limits and colimits in this category. More precisely: the initial object, the terminal object, the product, the coproduct, the equaliser, the coequaliser, the pullback, the pushout and the exponential.

In Section 3.2 we introduce two categories of fuzzy partitions, **Coverage** and **Partition**, which have the same objects but different morphisms, and we prove the following theorem:

Theorem 3.2.4. *The category* **Coverage** *is isomorphic to* $\mathbf{f}-$**Covering***, which is the full subcategory of* **Covering** *consisting in coverings with finitely many fuzzy sets.*

In Section 3.3, we introduce and study $\mathbf{n}-$**Covering**, the category of fuzzy normal coverings (coverings with fuzzy normal sets), and we prove the following result:

Theorem 3.3.4. *Let* $\mathfrak{I} : \mathbf{n-Covering} \to \mathbf{n-Covering}$ *be the functor defined:*

(a) On objects, $\mathfrak{I}\left((X,(A_i)_{i\in I})\right) = \left(I,(B_x)_{x\in X}\right)$, *where* $B_x(i) = A_i(x)$, $(\forall)x \in X$, $i \in I$.

(b) On morphisms, $\mathfrak{I}((f,\rho)) = (\rho, f)$.

Then $\mathfrak{I}$ *is an involution functor, i.e.* $\mathfrak{I}\circ\mathfrak{I} = 1_{\mathbf{n-Covering}}$.

In Section 3.4, we introduce and study the category of fuzzy tolerance relations.

Definition 3.4.1. *Let* **Tol** *be the category of fuzzy tolerance relations which has:*

(1) $\mathrm{Ob}(\mathbf{Tol}) = \{(X,T) \ : \ T : X\times X \to [0,1]$ *is a fuzzy tolerance relation*$\}$.

(2) $\mathrm{Hom}\left((X,T),(Y,S)\right) = \{f : X \to Y | T(x,y) \le S\left(f(x), f(y)\right), (\forall)x, y \in X\}$.

(3) If $f : (X,T) \to (Y,S)$ *and* $g : (Y,S) \to (Z,Q)$ *then their composition is* $g\circ f : (X,T) \to (Z,Q)$, $(g\circ f)(x) = g(f(x))$.

(4) $\mathrm{id}_{(X,T)} = \mathrm{id}_X$, *for all* $(X,T) \in \mathrm{Ob}(\mathbf{Tol})$.

As for the category **Covering**, we determine the limits and colimits in **Tol**, i.e. the initial object, the terminal object, the product, the coproduct, the equaliser, the coequaliser, the pullback, the pushout and the exponential. Moreover, we establish an isomorphism of categories between **Tol** and $\mathbf{t-Covering}$, the category of tolerance coverings, which is a full subcategory of **Covering**.

In the fourth chapter, using geometrical methods, we construct several isomorphisms between several subcategories of **Covering** and **Partition**. The central idea is to interpret a fuzzy covering of a set with n fuzzy set, as a map with values on the border of a n-dimensional cube, while a fuzzy partition can be interpreted as a map with values in a $(n-1)$-simplex.

Definition 4.1.2 *A covering* $(X,(A_i)_{i\in[n]})$ *of* X *is called a* good covering *if it satisfies the conditions* $\sum_{i\in[n]} A_i(x) \le n\cdot A_i(x) + 1$, *for all* $i \in [n]$ *and* $x \in X$.

We define the category $\mathbf{g-Covering}$ *as the full subcategory of* **Covering** *whose objects are good coverings.*

Theorem 4.1.3 *Let* $(B_i)_{i\in[n]}$ *be a partition of a set* X. *For* $n = 1$ *we let* $A_1 := B_1$. *For* $n \ge 2$, *we define* $(A_1(x),\ldots,A_n(x)) := \Phi(B_1(x),\ldots,B_n(x))$, *for all* $x \in X$, *where* Φ *is defined bellow:*

Then $(X,(A_i)_{i\in[n]})$ *is a good covering of* X. *Moreover, there is an isomorphism of categories, denoted for convenience* $\Phi : \mathbf{Partition} \to \mathbf{g-Covering}$, *defined:*

(1) On objects, $\Phi((X,(B_i)_{i\in[n]})) := (X,(A_i)_{i\in[n]})$, *where*

$$A_i(x) = B_i(x) + 1 - \bigvee_{i\in[n]} B_i(x), \text{ for all } i \in [n], x \in X.$$

(2) On morphisms, $\Phi((\rho, f)) := (\rho, f)$.

The inverse of Φ *is* $\Phi^{-1} : \mathbf{g - Covering} \to \mathbf{Partition}$ *and it is defined:*

(1) On objects, $\Phi^{-1}((X, (A_i)_{i\in[n]})) := (X, (B_i)_{i\in[n]})$*, where*

$$B_i(x) = A_i(x) - \frac{1}{n}\sum_{i\in[n]} A_i(x) + \frac{1}{n}, \text{ for all } i \in [n], x \in X.$$

(2) On morphisms, $\Phi((\rho, f)) := (\rho, f)$.

In Section 4.2, we establish an isomorphism between **Covering**$[n]$, the category of coverings with n fuzzy sets, and a subcategory of **Partition**, whose objects are partitions with n sets which satisfies certain conditions.

Let $\mathcal{P}$ be the convex hull of

$$\left\{\left(\frac{1-s}{n} + \alpha_1, \ldots, \frac{1-s}{n} + \alpha_n\right) \;:\; \alpha_i \in \{0, 1\},\ s = \alpha_1 + \cdots + \alpha_n\right\}$$

We consider **Partition**$[n]_{\mathcal{P}P}$, the full subcategory of **Partition**$[n]$, whose objects are $(X, (B_1, \ldots, B_n))$ such that $(B_1(x), \ldots, B_n(x)) \in \mathcal{P}$ for all $x \in X$.

Definition 4.2.10. *We define the functor* $\mathbf{F[n]} : \mathbf{Covering}[n] \to \mathbf{Partition}[n]_{\overline{\mathcal{P}}}$*, as follows:*

(1) On objects: $\mathbf{F[n]}((X, (A_i)_{i\in[n]})) := (X, (B_i)_{i\in[n]})$*, where*

$$B_i(x) = \frac{1}{n-1} A_i(x) - \frac{1}{n(n-1)}\left(\sum_{i\in[n]} A_i(x)\right) + \frac{1}{n} \text{ for all } i \in [n], x \in X.$$

(2) On morphisms: $\mathbf{F[n]}((f, \rho)) := (f, \rho)$.

We also define the functor $\mathbf{G[n]} : \mathbf{Partition}[n]_{\overline{\mathcal{P}}} \to \mathbf{Covering}[n]$*, as follows:*

(1) On objects: $\mathbf{G[n]}((X, (B_i)_{i\in[n]})) := (X, (A_i)_{i\in[n]})$*, where*

$$A_i(x) = (n-1)(B_i(x) - \bigvee_{i\in[n]} B_i(x)) + 1 \text{ for all } i \in [n], x \in X.$$

(2) On morphisms: $\mathbf{G[n]}((f, \rho)) := (f, \rho)$.

Theorem 4.2.11. *With the above notations, the functors* $\mathbf{F[n]}$ *and* $\mathbf{G[n]}$ *are well defined and and fully faithful. Moreover, they induce an isomorphism of categories between* **Covering**$[n]$ *and* **Partition**$[n]_{\overline{\mathcal{P}}}$.

The cases $n = 2$ and $n = 3$ are presented with more details in Section 4.3, and Section 4.4 respectively.

Conclusions

The thesis introduce a new and fruitful approach, using category theory, in the study of fuzzy coverings and fuzzy partition, opening new directions of research in this area. The main contributions of this thesis are related to the introduction and the study of several categories of fuzzy coverings, fuzzy tolerance relations and fuzzy partitions and the relation between them. These results were published in four ISI articles ([14], [15], [5] and [16]).

Acknowledgements

The author is deeply indebted to his advisers Professor Paul Flondor and Professor Mircea Cimpoeaş for all of their help regarding the conception and writing of this thesis.

The author owes a sincerely thank to his family for their invaluable support.

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