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Operators with fixed point in JS generalized metric spaces

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Abstract of the PhD Thesis

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Contents

Abstract

General objective. Fixed point theory proved to be an useful tool in solving problems arising in branches of modern sciences, as many real world phenomena may be modeled by various types of equations whose solutions can be reckoned by associating a fixed point problem. One direction of development of such theories refers to the contractive conditions which are imposed to the operators in question, in order to ensure that they possess a fixed point, and additionally, that the uniqueness of the fixed point is accomplished. Furthermore, the underlying space plays an important role in designing fixed point theorems. The main aim of this PhD thesis is to develop results on the existence and uniqueness of fixed point(s) associated to operators with suitable properties in a general setting. More precisely, the environment used here is the generalized metric space introduced by Jleli and Samet [19], by changing the first and the third axiom of the metric. As the family of these extended metric spaces contains, apart from the classic metric spaces, other generalized classes, such as b-metric spaces, dislocated metric spaces or modular spaces which satisfy the Fatou property, known results in literature are generalized by the theorems stated here. Regarding the outcomes presented in this thesis, they refer to the study of operators which are defined by various types of weakly contractive conditions, such as: convex contractive mappings, operators which satisfy inequalities defined by quantities which contain rational terms, mappings which check conditions introduced by the use of auxiliary functions with adequate properties, and operators of Geraghty type with additional properties meant to overcome the lack of any triangle-type inequality in the framework of the Jleli-Samet metric spaces.

Methodology. The methods used to develop the results in this thesis are specific to the fixed point theory, particularly to the case of the framework provided by the Jleli-Samet metric spaces. The main difficulty which had to be overcome was the absence of any kind of a triangle type inequality, which is usually the main tool to prove the property of a sequence to be a Cauchy one. This problem was solved, on one hand, by imposing adequate boundedness conditions with respect to the orbit of the mappings

studied, about a certain element in the generalized metric space. Moreover, a part of the theorems regarding the existence and uniqueness of the fixed points were stated with the help of adequate preorders and monotone or continuity properties associated with them. The problems of proving that the limit of the Picard sequence is indeed a fixed point were solved by enlarging the set of points on which the contractive relation is checked, or by imposing conditions related to types of continuity on the operators.

General state of art. Fixed point theory is a topical field of research in modern mathematics, one of the motives being that it provides a helpful tool in solving various types of equations, some of which arise from practical reasons. These equations may be arranged into a form in which their solution is precisely the fixed point of the involved mapping. The first question to be answered is if there exists a fixed point of the involved mapping, and then if such a point is a unique one. The contraction principle is one of the most renowned results in this direction, regarding mappings which diminish the distance between any two points after their application; for the pioneer source, please see Caccioppoli [6]. Edelstein [14] proved that, in the framework of a compact metric space, the condition that the distance between the images of any two points is strictly smaller that the distance between the precise two points ensures the existence of a fixed point of the operator. These conditions proved to be sometimes too restrictive, since all such operators have to be continuous, fact which led to various extensions of this concept. Kannan [22] changed the form of the contractive inequality by using the image of the points in both parts of it, obtaining a class of mappings which posses a fixed point which is unique, but are not necessarily continuous. Reich [29] unified the contraction principle and the Kannan operators in a more general contractive condition. Sehgal [33] improved the idea of Edelstein [14], by taking into consideration the maximum of the distances in the contraction principle and Kannan theorem, and a continuous operator. Reich [30] extended in a new fashion the idea of weakly contractivity, by means of multiplication of the various distances appearing in the contractive inequality by non-increasing functions which additionally satisfy a boundedness type condition with respect to their sum. Bianchini [4] replaced the sum used by Kannan by a term which is defined as the maximum of the distances between the considered points and their images through this mapping, multiplied by a subunitary value. The idea of a different combination of distances than that of Kannan was pointed out by Chatterjea [7].

Metric spaces have been an adequate framework for important outcomes, in various fields of mathematics, including fixed point theory. Extensions of the classic metric spaces have proved to be useful both from theoretical and applied point of view. Hitzler and Seda [17] changed the first of its axioms in order to introduce the dislocated metric spaces. Dropping the second condition regarding the symmetry from the metric definition was another way to extend this concept, by Wilson [35]. A fruitful method to generalize metric spaces is related to the triangle inequality. In this respect, we refer here, for example, to b-metric spaces introduced by Czerwik [8] and Bakhtin [3], in which the right hand side of the triangle inequality from the metric space was multiplied by a constant bigger that one. This change had a powerful effect, since the b-metrics are no longer continuous functions. Still the property of uniqueness of the limit of a convergent sequence stays on. Branciari [5] modified the triangle inequality into a quadrilateral one, obtaining the rectangular metric spaces; here the uniqueness of a limit of a convergent sequence does no longer hold. Kamran *et al.* [21] introduced the extended b-metric spaces, by multiplying the right hand side from the triangle inequality by an adequate function. Another extension of metric spaces is the modular space by Nakano [28] and then Musielak and Orlicz [27]. Partial metric spaces were introduced by Matthews [25], from reasons related to theoretic computer science. Various fixed-point results have been developed in these metric structures. The setting of dislocated metrics was used by Jungck and Rhoades [20] to study weakly compatible maps, while Hitzler [16] connected them to real world applications in logic programming semantics. Samreen *et al.* [31] adapted comparison functions to the context of extended b-metric spaces in order to design generalized contractive conditions.

In [19], Jleli and Samet introduced a generalization of classic metric spaces, which strictly includes some of the extensions previously discovered, such as the b-metric spaces, the dislocated metric spaces or the modular spaces which satisfy the Fatou property. In the same work, they prove contractive principles in this setting. Karapınar *et al.* [24] used a binary relation with the aim of developing some existence results in this generalized context, which was also considered by Senapati *et al.* [34], to extend Wardowski implicit contractive conditions. Altun and Samet [2] had in view pseudo-Picard operators in the framework of Jleli-Samet metric spaces, which were also used by Karapınar *et al.* [23] to state Meir-Keeler type results. Sawangsup and Sintunavarat [32] approached the topic of transitive relations from the point of view of defining suitable operators in these spaces.

Thesis description: structure and content.

Chapter 1 has an introductory purpose, it presents notions and concepts which are needed to develop the results in the next chapters. They mainly refer to the setting in which the theorems are developed, namely the Jleli-Samet metric spaces.

Definition 1. Let us consider the arbitrary set $X \neq \emptyset$ and let $D\colon X \times X \to [0,\infty]$ be a mapping. For each point $x_* \in X$, we have in view the set of all sequences $\{x_n\} \subset X$ from the set

$$\mathcal{C}(D,X,x_*) = \big\{\{x_n\} \subset X : \lim_{n\to\infty} D(x_n,x_*) = 0\big\}.$$

We say that D is a $JS-$metric on X if the following axioms are satisfied:

$(D1)$ For all x, $y \in X$, the equality

$$D(x,y) = 0 \text{ implies } x = y;$$

$(D2)$ For every x, $y \in X$, the symmetry condition

$$D(x,y) = D(y,x)$$

is satisfied;

$(D3)$ There is a constant $C > 0$ such that for every x, $y \in X$, and for each sequence $\{x_n\} \in \mathcal{C}(D,X,x)$, the following inequality is valid

$$D(x,y) \leq C \limsup_{n\to\infty} D(x_n,y).$$

In our study, the pair (X,D) will always denote a Jleli-Samet metric space (also called a JS-space), with the exception of the case when it is clearly mentioned otherwise.

With respect to the relations that JS-metric spaces have with the spaces presented previously, as proved in [19], this class comprises that of classic metric spaces, b-metric spaces, dislocated metric spaces, or modular spaces with the Fatou property. On the other hand, it is wider than the reunion of these classes. In this respect, we address the reader, for example, to [24] and the references therein. Let us give an illustrative example in this direction.

Example 1. Let $X = \{0,p,q\}$ with p, $q \in \mathbb{R}^*$, $p \neq q$, and

$$D\colon X \times X \to [0,\infty], \qquad \begin{cases} D(0,0) = 0; \\ D(0,p) = D(p,0) = D(p,p) = p; \\ D(0,q) = D(q,0) = D(p,q) = D(q,p) = q; \\ D(q,q) = \infty. \end{cases}$$

This example shows that there are Jleli-Samet spaces that do not belong to any classes of metrics above mentioned.

Definition 2. Let (X,D) be a JS-metric space and consider $\{x_n\}$ a sequence in X.

1. We say that $\{x_n\}$ is convergent to $x \in X$ if

$$\lim_{n\to\infty} D(x_n, x) = 0.$$

2. It will be said that $\{x_n\}$ is a D-Cauchy sequence if

$$\lim_{m,n\to\infty} D(x_m, x_n) = 0.$$

As in the classical case, it is immediate that the limit of a convergent sequence is unique. Moreover, it is worth mentioning that any convergent sequence in the JS-metric is also D-Cauchy; the converse is no longer true. A JS-space (X, D) in which each D-Cauchy sequences is D-convergent to an element in X is called D-complete.

Along the Thesis, we are going to use the next notations. Define

$$\delta_{n_0}(D, T, x) = \sup(\{D(T^n x, T^m x) : n, m \in \mathbb{N}, n, m \geq n_0\}),$$

where $n_0 \in \mathbb{N}$ is an index, and

$$\delta(D, T, x) = \sup(\{D(T^n x, T^m x) : n, m \in \mathbb{N}\}).$$

Moreover, we denote the orbit of an element x by an operator $T\colon X \to X$ as

$$\mathcal{O}_T(x) = \{T^n x : n \in \mathbb{N}\}.$$

Definition 3. For a given preorder $\mathbf{B}$, a sequence $\{x_n\} \subset X$ is $\mathbf{B}$-nondecreasing if $x_n \mathbf{B} x_{n+1}$, for all $n \in \mathbb{N}$.

Closely related to the preorder $\mathbf{B}$ are the adequate types of monotone regularity, as follows.

Definition 4. The Jleli-Samat metric space (X, D) is $\mathbf{B}$-nondecreasing-regular, where $\mathbf{B}$ is a preorder, if for every $\{x_n\} \in \mathcal{C}(D, X, z)$ that is $\mathbf{B}$-nondecreasing it happens that $x_n \mathbf{B} z$, for all $n \in \mathbb{N}$.

The monotone of operators will be needed along some parts of this work.

Definition 5. Let (X, D) be a Jleli-Samet space endowed with a binary relation $\mathbf{B}$, which is also a preorder, and $T\colon X \to X$. The operator T is called B-nondecreasing if $x\mathbf{B}y$ implies $Tx\mathbf{B}Ty$ for all x, $y \in X$.

Definition 6. The Jleli-Samat metric space (X, D) is $\mathbf{B}$-nondecreasing-complete if every $\{x_n\}$ which is D-Cauchy and $\mathbf{B}$-nondecreasing is D-convergent in X.

Denote by Ψ the family of functions $\psi\colon [0,\infty) \to [0,\infty)$ which are upper semi-continuous, strictly nondecreasing, and for which $\varphi(t) < t$, for each $t > 0$. Note that such functions necessarily check the equality $\varphi(0) = 0$. Other set which is used frequently in this work is Θ, which contains all continuous functions $\theta\colon [0,+\infty)^4 \to [0,+\infty)$, with the properties that $\theta(0,t,s,u) = 0$, and $\theta(t,s,0,u) = 0$, for all t, s, $u \in [0,+\infty)$. Another tool used in the formulation of a part of the extensions of fixed point theorems which will be presented is the so-called α-admissibility.

Definition 7. Let us consider $X \neq \emptyset$, and the mapping $\alpha\colon X \times X \to [0,\infty)$. An operator $T\colon X \to X$ is α-admissible if $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$, for all x, $y \in X$.

Triangular admissibility, and α-regularity have been applied in order to prove some of the results in this Thesis.

Definition 8. Let us consider $X \neq \emptyset$ and the mapping $\alpha\colon X \times X \to [0,\infty)$. We say that $T\colon X \to X$ is a triangular α-admissible mapping if the inequality $\alpha(x,y) \geq 1$ implies that $\alpha(Tx,Ty) \geq 1$, and the relations $\alpha(x,y) \geq 1, \alpha(y,z) \geq 1$ imply $\alpha(x,z) \geq 1$, for all x, y, $z \in X$.

Definition 9. Let (X,D) be a Jleli-Samet space and $\alpha\colon X \times X \to [0,\infty)$. (X,D) is called JS α-regular if for any sequence $\{x_n\}$ convergent to x and $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, there is a subsequence of the initial sequence such that $\alpha(x_{n_k},x) \geq 1$, for all $k \in \mathbb{N}$.

Chapter 2 titled **Convex contractive mappings** [13] develops a study of convex contractive mappings in the setting of JS-metric spaces. The notion of a convex contraction was introduced by Istrăţescu in his work [18]. In this paper, Istrăţescu proved some existence and uniqueness results regarding various types of convex contractions. More precisely, the right hand side of the contractive inequality consists in combinations of distances and the scalars involved have their sum smaller than one. Changing the distances or adding new terms in the inequality brings out new types of contractions to be studied. The ideas were extended successfully in various spaces and the convex inequality has been improved by the use of additional suitable terms. These outcomes were further developed by Miandaragh *et al.* [26], who stated theorems of fixed points in metric spaces by means of additional properties of the studied operators. More explicitly, they use the notion of α-admissibility and the property (H) in order to establish the existence and the uniqueness of a fixed point for operators with adequate properties.

This Chapter glue up together all these tools in order to extend the theory in the settings of generalized spaces.

Definition 10. Let $T\colon X \to X$ be a self mapping. For $\varepsilon > 0$, we say that $x_* \in X$ is an ε-fixed point of T if $D(x_*, Tx_*) < \varepsilon$. As a notation, $F_\varepsilon(T)$ is the set of all ε-fixed points of T. Moreover, it is said that T has the the approximate fixed point property if for any $\varepsilon > 0$, T has an ε-fixed point.

Definition 11. Let us consider $X \neq \emptyset$ and the mapping $\alpha\colon X \times X \to [0, \infty)$. We say that X has the property (H) if, for each x, $y \in X$ there is a point $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

In practice, working with general Jleli-Samet metric spaces is problematic when it comes to establish properties related to fixed points. A way to avoid problematic situations is to restrict the values of the metric such that pathological cases with infinite distances do not appear.

Definition 12. If (X, D) is a Jleli-Samet metric space such that $D(x, y) < \infty$, for all x, $y \in X$, we say that (X, D) is a strong Jleli-Samet metric space.

Theorem 1. *Let (X, D) be a Jleli-Samet metric space, $T\colon X \to X$ be an operator and $\alpha\colon X \times X \to [0, \infty)$ a given mapping. Suppose that the following conditions are fulfilled:*

i) *T is α-admissible;*

ii) *there is $x_0 \in X$ with $\delta(D, T, x_0) < \infty$, such that $\alpha(x_0, Tx_0) \geq 1$;*

iii) *there are a, $b \in [0, 1)$ with $a + b < 1$, such that*

$$\alpha(x, y)D(T^2x, T^2y) \leq aD(Tx, Ty) + bD(x, y),$$

for all x, $y \in \mathcal{O}_T(x_0)$;

Then the mapping T has the approximate fixed point property.

Theorem 2. *Let (X, D) be a complete Jleli-Samet metric space, $T\colon X \to X$ be an operator and $\alpha\colon X \times X \to [0, \infty)$ a given mapping. Suppose that the following conditions are fulfilled:*

i) *T is a triangular α-admissible mapping;*

ii) *there is $x_0 \in X$ with $\delta(D, T, x_0) < \infty$, such that $\alpha(x_0, Tx_0) \geq 1$;*

iii) *there are a, $b \in [0, 1)$ with $a + b < 1$, such that*

$$\alpha(x, y)D(T^2x, T^2y) \leq aD(Tx, Ty) + bD(x, y),$$

for all x, $y \in \mathcal{O}_T(x_0)$;

iv) T *is continuous;*

v) $\alpha(T^n x_0, T^n x_0) \geq 1$, *for all* $n \in \mathbb{N}$.

Then the sequence $\{T^n x_0\}$ *is convergent to a point* x_* *in* X. *Moreover,* x_* *is a fixed point of* T.

As a straight forward remark, if there are two fixed points of T, forming the pair (x_*, y_*), at which α is not smaller than one and the distance between them is finite, and the contractive inequality holds, then the uniqueness property of the fixed point follows.

In order to state a result of the uniqueness of fixed points of mappings which fulfill the convexity condition, regardless of the value of the function α at the pair formed by two presumed fixed points, we have to impose an additional property to these operators, but instead we are allowed to remove some of those already used.

Theorem 3. *Let* (X, D) *be a strong Jleli-Samet metric space and* $T\colon X \to X$ *be a self-mapping. Presume that the following assertions hold true:*

i) T *is* α*-admissible for some* $\alpha\colon X \times X \to [0, \infty)$;

ii) *there are* a, $b \in [0, 1)$ *with* $a + b < 1$, *such that*

$$\alpha(x, y)D(T^2x, T^2y) \leq aD(Tx, Ty) + bD(x, y),$$

for all x, $y \in X$;

iii) T *has property* (H);

Then, if T *has a fixed point, it is unique.*

Another series of results can be established following the notions introduced in [26], as the concept of a two sided generalized convex contraction, as follows.

Theorem 4. *Let* (X, D) *be a Jleli-Samet metric space,* $T\colon X \to X$ *be an operator and* $\alpha\colon X \times X \to [0, \infty)$ *a given mapping. Presume that the following assertions are accomplished:*

i) T *is* α*-admissible;*

ii) *there is* $x_0 \in X$ *with* $\delta(D, T, x_0) < \infty$, *such that* $\alpha(x_0, Tx_0) \geq 1$;

iii) *there are* a_1, a_2, b_1, $b_2 \in [0, 1)$ *with* $a_1 + a_2 + b_1 + b_2 < 1$, *such that*

$$\begin{aligned}\alpha(x, y)D(T^2x, T^2y) \leq\ & a_1D(x, Tx) + a_2D(Tx, T^2x)\\ & +b_1D(y, Ty) + b_2D(Ty, T^2y),\end{aligned}$$

for all x, $y \in \mathcal{O}_T(x_0)$;

Then the mapping T *has the approximate fixed point property.*

Theorem 5. *Let (X, D) be a complete Jleli-Samet metric space, $T\colon X \to X$ be an operator and $\alpha\colon X \times X \to [0,\infty)$ a given mapping. Suppose that the following conditions are fulfilled:*

i) *T is triangular α-admissible;*

ii) *there is $x_0 \in X$ with $\delta(D, T, x_0) < \infty$, such that $\alpha(x_0, Tx_0) \geq 1$;*

iii) *there are a_1, a_2, b_1, $b_2 \in [0, 1)$ with $a_1 + a_2 + b_1 + b_2 < 1$, such that*

$$\begin{aligned}\alpha(x,y)D(T^2x,T^2y) \leq{}& a_1D(x,Tx) + a_2D(Tx,T^2x)\\ &+b_1D(y,Ty) + b_2D(Ty,T^2y),\end{aligned}$$

for all x, $y \in \mathcal{O}_T(x_0)$;

iv) *T is continuous;*

v) *$\alpha(T^nx_0, T^nx_0) \geq 1$, for all $n \in \mathbb{N}$.*

Then the sequence $\{T^nx_0\}$ is convergent to x_*. Moreover, x_* is a fixed point of T.

In Chapter 3 **Rational contractive conditions** [10] we extend the rational type contractive operators in the context of JS-spaces. In 2015, Alsulami *et al.* [1] developed the notion of (α, ψ)-rational type contraction, which is a mixture of mappings with different properties, connected by an inequality condition. Their study was made in the context of generalized metric spaces, where the third axiom is an extension of the classical one by the addition of a new term, and therefore the triangle-type inequality plays a crucial role in proving some existence results for fixed points. Another step in the study of new suitable weak contractions was achieved by Wu and Zhao in [36], where they introduced the (α, ψ)-rational type contractions in the setting of b-metrics. The importance of the rational contractions led to extensions of the theory in spaces which possess different structures compared with the classical ones. By a combination of technical ideas and an adequate methodology, fixed point results will be proved by taking into account the restrictions imposed by working without the benefits of any kind of triangle inequality. Our approach is different than that in Alsulami *et al.* [1], where a generalized metric in which a quadrilateral inequality holds is used (in these spaces, the uniqueness of the limit of a convergent sequence is not guaranteed).

Theorem 6. *Let (X, D) be a complete Jleli-Samet metric space, $T\colon X \to X$ be an operator and $\alpha\colon X \times X \to [0,\infty)$ be a given mapping. Suppose that the following conditions are fulfilled:*

i) *T is a triangular α-admissible mapping;*

ii) *there is* $x_0 \in X$ *with* $\delta(D, T, x_0) < \infty$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*

iii) *there is a function* $\psi \in \Psi$ *such that:*

$$\alpha(x, y)D(Tx, Ty) \leq \psi(M(x, y)),$$

where

$$\begin{aligned} M(x,y) \quad &= \quad \max\Big\{ D(x,y), D(x,Tx), \frac{D(x,Tx)D(y,Ty)}{1+D(x,y)}, \\ &\qquad \frac{D(x,Tx)D(y,Ty)}{1+D(Tx,Ty)} \Big\}, \end{aligned}$$

for all x*,* $y \in \mathcal{O}'_T(x_0) = \mathcal{O}_T(x_0) \cup \{\omega \in X : \lim_{n\to\infty} D(T^n x_0, \omega) = 0\}$*;*

iv) X *is* α*-regular;*

v) $D(T^n x_0, T^n x_0) = 0$*, for all* $n \in \mathbb{N}$*.*

Then the sequence $\{T^n x_0\}$ *is convergent to a point* $x_* \in X$*. Moreover, if* $D(x_*, Tx_*) < \infty$*, then* x_* *is a fixed point of* T*.*

In addition, if there is another fixed point of T*, denoted by* y_**, with* $D(y_*, y_*) < \infty$*,* $\alpha(x_*, y_*) \geq 1$ *and* $D(x_*, y_*) < \infty$*, so that the pair* (x_*, y_*) *checks hypothesis* iii)*, then* $x_* = y_*$*.*

In the next theorem, we weaken the contractive condition by adding an additional term in the right hand side, and for this reason we need another condition instead of that in iv).

Theorem 7. *Let* (X, D) *be a complete Jleli-Samet metric space,* $T \colon X \to X$ *be an operator and* $\alpha \colon X \times X \to [0, \infty)$ *be a given mapping. Suppose that the following items are accomplished:*

i) T *is a triangular* α*-admissible mapping;*

ii) *there is* $x_0 \in X$ *with* $\delta(D, T, x_0) < \infty$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*

iii) *there is a mapping* $\psi \in \Psi$ *such that:*

$$\alpha(x, y)D(Tx, Ty) \leq \psi(M(x, y)),$$

where

$$\begin{aligned} M(x,y) \quad &= \quad \max\Big\{ D(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Tx)D(y,Ty)}{1+D(x,y)}, \\ &\qquad \frac{D(x,Tx)D(y,Ty)}{1+D(Tx,Ty)} \Big\}, \end{aligned}$$

for all x*,* $y \in \mathcal{O}_T(x_0)$*;*

iv) *T is continuous;*

v) *$D(T^n x_0, T^n x_0) = 0$, for all $n \in \mathbb{N}$.*

Then T has a fixed point $x_* \in X$ and the Picard sequence $\{T^n x_0\}$ goes to x_*. In addition, if there is another fixed point of T, denoted by y_*, with $D(y_*, y_*) = 0$, $\alpha(x_*, y_*) \geq 1$, and $D(x_*, y_*) < \infty$, so that (x_*, y_*) checks the contractive inequality, then $x_* = y_*$.

Example 2. Let us consider the set $A = \{0, 1, 2, \ldots, N\}$, where $N \geq 2$, endowed with the Jleli-Samet metric $D\colon A \times A \to [0.\infty]$ defined in the following way:

$$D(x,y) = \begin{cases} 0, & \text{if } x = y \text{ and } x, y \in \{0, 1, 2, \cdots, N-1\}, \\ x + y, & \text{if } x \neq y \text{ and } x, y \in \{0, 1, 2, \cdots, N\}, \\ \infty, & \text{if } x = y = N. \end{cases}$$

Let us take $T\colon A \to A$ given by $T(0) = T(1) = \cdots = T(N-1) = 1$ and $T(N) = N - 1$. Consider then $\psi\colon [0, \infty) \to [0, \infty)$, $\psi(t) = \frac{2}{3}t$, and $\alpha\colon A \times A \to [0, \infty)$ with $\alpha(x, y) = 1$, for all x, $y \in A$. All conditions of Theorem 6 are satisfied, and the fixed point is $x_* = 1$.

By changing the form of the term $M(x, y)$, we obtain other interesting fixed point results in this setting.

For such a class of operators, the following theorem of existence and uniqueness that can be proved.

Theorem 8. *Let (X, D) be a complete JS space, $T\colon X \to X$ be an operator and $\alpha\colon X \times X \to [0, \infty)$ be a given function. Suppose that the following constraints are satisfied:*

i) *T is a triangular α-admissible mapping;*

ii) *there is $x_0 \in X$ with $\delta(D, T, x_0) < \infty$ such that $\alpha(x_0, Tx_0) \geq 1$;*

iii) *there is a mapping $\psi \in \Psi$ such that:*

$$\alpha(x, y) D(Tx, Ty) \leq \psi(M(x, y)),$$

where

$$\begin{aligned} M(x,y) \;=\; & \max\Bigg\{ D(x,y), D(x,Tx), \frac{D(x,Tx)D(x,T^2x)}{1 + D(x,Ty)}, \\ & \frac{D(x,Tx)D(y,Ty)}{1 + D(y,T^2x)} \Bigg\}, \end{aligned}$$

for all x, $y \in \mathcal{O}'_T(x_0)$;

iv) *X is α-regular;*

v) *$D(T^n x_0, T^n x_0) = 0$ for all $n \in \mathbb{N}$.*

Then $\{T^n x_0\}$ is convergent to a point $x_* \in X$. Moreover, if $D(x_*, Tx_*) < \infty$, then x_* is a fixed point of T.

Additionally, if there is another fixed point of T, denoted by y_* with $D(y_*, y_*) < \infty$, $\alpha(x_*, y_*) \geq 1$ and $D(x_*, y_*) < \infty$, so that (x_*, y_*) is fulfilling the contractive inequality, then $x_* = y_*$.

In Chapter 4, **Weakly contractive mappings** [12], the principal motivation is the study of generalized contractive operators defined by means of comparison type functions and continuous mappings of four variables in the context of Jleli-Samet metric spaces. These mappings are additionally endowed with adequate conditions which allow to overcome the lack of a classic triangle kind inequality. Here the focus is on conditions inspired by the classic contraction principle, Kannan type contractions or those from the Ćirić class, which led to existence and uniqueness fixed point outcomes in these generalized metric spaces. The working JS spaces satisfy monotone and/or regularity assumptions with respect to a preorder relation. Moreover, the mappings studied from these points of view fulfill adequate types of continuity.

Theorem 9. Let (X, D) be a Jleli-Samet metric space and let $T \colon X \to X$ be a self-mapping. Presume that the next conditions are fulfilled:

i) There is $x_* \in X$, and $n_0 \in \mathbb{N}$ for which we have $\delta_{n_0}(D, T, x_*) < \infty$;

ii) There is $\varphi \in \Psi$ and $\theta \in \Theta$ such that:

$$D(Tx, Ty) \leq \varphi(D(x, y)) + \theta(D(y, Tx), D(x, Ty), D(x, Tx), D(y, Ty)),$$

for all x, $y \in \mathcal{O}_T(x_*)$;

iii) $D(T^n x_*, T^n x_*) = 0$, for all $n \in \mathbb{N}$.

Then, the Picard sequence $\{T^n x_*\}$ is D-Cauchy.

Theorem 10. *Let (X, D) be a Jleli-Samet metric space endowed with a preorder $\mathbf{B}$ such that it is $\mathbf{B}$-nondecreasing-complete and $T \colon X \to X$ a mapping. Presume that the next conditions are fulfilled:*

i) *There is $x_* \in X$ such that $x_* \mathbf{B} T x_*$, and $n_0 \in \mathbb{N}$ for which $\delta_{n_0}(D, T, x_*) < \infty$;*

ii) *There is $\varphi \in \Psi$, and $\theta \in \Theta$, such that:*

$$D(Tx, Ty) \leq \varphi(D(x, y)) + \theta(D(y, Tx), D(x, Ty), D(x, Tx), D(y, Ty)),$$

for all x, $y \in \mathcal{O}_T(x_*)$;

iii) *$D(T^n x_*, T^n x_*) = 0$, for all $n \in \mathbb{N}$;*

iv) *T is $\mathbf{B}$-nondecreasing, and $\mathbf{B}$-nondecreasing-continuous.*

Then, the Picard sequence $\{T^n x_*\}$ is D-convergent to a fixed point ω of T, and $D(\omega, \omega) = 0$.

Moreover, if there is another fixed point of T, denoted by ω', so that (ω, ω') fulfills ii)*, then if $D(\omega, \omega') < \infty$ and $D(\omega', \omega') < \infty$, then $\omega = \omega'$.*

The next example illustrates the application of these outcomes.

Example 3. Let us start with the set $X = [0, 1]$ endowed with the metric

$$D(x, y) = \begin{cases} 0, & x = y; \\ \max\{x, y\}, & x \neq y. \end{cases}$$

(X, D) is a complete Jleli-Samet space, on which we define the operator

$$T\colon X \to X, \quad Tx = x - \ln(1 + x).$$

Consider now the comparison function

$$\varphi\colon [0, \infty) \to [0, \infty), \quad \varphi(x) = \begin{cases} \dfrac{x^2}{2} - \dfrac{x^3}{3}, & \text{if } x \leq 1, \\ \dfrac{x}{6}, & \text{if } x > 1, \end{cases}$$

and $\theta(t, u, v, w) = \frac{3}{2} v^2 t$, $\theta \in \Theta$. T is (φ, θ)-contraction and has a unique fixed point, $x = 0$.

Generalizations in the sense of Kannan and Ćirić can also be formulated and proved.

Theorem 11. *Let (X, D) be a Jleli-Samet metric space which is $\mathbf{B}$-nondecreasing-complete relative to the preorder $\mathbf{B}$ and $T\colon X \to X$ a mapping. Presume that:*

i) *There exists $x_* \in X$ such that $x_* \mathbf{B} T x_*$, and $\delta_{n_0}(D, T, x_*) < \infty$ for some $n_0 \in \mathbb{N}$;*

ii) *There is $\varphi \in \Psi$ and $\theta \in \Theta$ such that:*

$$\begin{aligned} D(Tx, Ty) \ &\leq \ \varphi\left(\frac{D(x, Tx) + D(y, Ty)}{2}\right) \\ &\quad + \theta(D(y, Tx), D(x, Ty), D(x, Tx), D(y, Ty)), \end{aligned}$$

for all x, $y \in \mathcal{O}_T(x_*)$;

iii) *$D(T^n x_*, T^n x_*) = 0$, for all $n \in \mathbb{N}$;*

iv) *T is $\mathbf{B}$-nondecreasing and $\mathbf{B}$-nondecreasing-continuous.*

Then, $\{T^n x_*\}$ is D-convergent to a fixed point ω of T, and $D(\omega, \omega) = 0$. Moreover, if ω' is a fixed point of T so that (ω, ω') checks hypothesis ii)*, for which $D(\omega, \omega') < \infty$, and $D(\omega', \omega') = 0$, then $\omega = \omega'$.*

Theorem 12. *Let (X, D) be a Jleli-Samet metric space that is $\mathbf{B}$-nondecreasing-complete relative to the preorder $\mathbf{B}$ and $T\colon X \to X$ an operator. Suppose that the next hypotheses are fulfilled:*

i) *There is $x_* \in X$ such that $x_* \mathbf{B} T x_*$ and $\delta_{n_0}(D, T, x_*) < \infty$ for some $n_0 \in \mathbb{N}$;*

ii) *There is $\varphi \in \Psi$ and $\theta \in \Theta$ such that:*

$$\begin{aligned} D(Tx, Ty) \quad &\leq \quad \varphi\left(\max\{D(x, y), D(x, Tx), D(y, Ty)\}\right) \\ &\quad\quad + \theta(D(y, Tx), D(x, Ty), D(x, Tx), D(y, Ty)), \end{aligned}$$

for all x, $y \in \mathcal{O}_T(x_*)$;

iii) *$D(T^n x_*, T^n x_*) = 0$, for all $n \in \mathbb{N}$;*

iv) *T is $\mathbf{B}$-nondecreasing and $\mathbf{B}$-nondecreasing-continuous.*

Then, the Picard sequence $\{T^n x_*\}$ is D-convergent to a point ω of T, and $D(\omega, \omega) = 0$. If ω' is another fixed point of T so (ω, ω') checks the inequality from ii)*, with $D(\omega, \omega') < \infty$ and $D(\omega', \omega') = 0$, then $\omega = \omega'$.*

Chapter 5, **Geraghty type operators** [9, 11], is devoted to weakly Geraghty type contractions, in the framework of Jleli and Samet. Various results are obtained, by combining the classic Geraghty class with adequate expressions, and summing the resulting operator by a continuous functions with additional properties. The initial idea of Geraghty [15] was further developed by weakening the condition which has to be checked by the operators, by adding adequate expressions related to continuous functions which satisfy also other hypotheses. Passing from the usual context of metric spaces to that of those developed by Jleli and Samet compelled the introduction of various additional conditions on the metric function or on the operators themselves. Some of the original results are obtained by using adequate preorders or continuity assumptions, other by enlarging the set of points on which the contractive relation is accomplished.

Denote the set $\mathbf{S}$ as the class of functions $\beta\colon [0, \infty) \to [0, 1)$ such that for any given sequence $\{t_n\} \subset [0, \infty)$, for which $\lim_{n\to\infty} \beta(t_n) = 1$, it follows that $\lim_{n\to\infty} t_n = 0$.

Theorem 13. *Let (X, D) to be a Jleli-Samet metric space and let $T\colon X \to X$ be a self-operator. Presume that the next conditions are fulfilled:*

i) *There is $\tilde{x} \in X$, and $n_0 \in \mathbb{N}$ for which we have $\delta_{n_0}(D, T, \tilde{x}) < \infty$;*

ii) *There is a Geraghty-type function $\beta \in \mathbf{S}$ and $\theta \in \Theta$ such that:*

$$\begin{aligned} D(Tx, Ty) \leq &\beta(D(x, y))D(x, y) \\ &+ \theta(D(y, Tx), D(x, Ty), D(x, Tx), D(y, Ty)), \end{aligned} \tag{1}$$

for all $x,\, y \in \mathcal{O}_T(\tilde{x})$*;*

iii) *The mapping defined by*

$$g\colon [0,\infty) \to [0,\infty), \quad g(t) = \beta(t)t,$$

is nondecreasing;

iv) $D(T^n\tilde{x}, T^n\tilde{x}) = 0$*, for all* $n \in \mathbb{N}$*.*

Then, the Picard sequence $\{T^n\tilde{x}\}$ *is D-Cauchy.*

Theorem 14. *Consider* (X, D) *to be a Jleli-Samet space for which there is a preorder* $\mathbf{A}$ *such that the space is* $\mathbf{A}$*-nondecreasing-complete. Let* $T\colon X \to X$ *be a mapping and suppose that the following conditions are satisfied:*

i) *There is* $\tilde{x} \in X$*, and* $n_0 \in \mathbb{N}$ *for which we have* $\delta_{n_0}(D, T, \tilde{x}) < \infty$*;*

ii) *There is a Geraghty-type function* $\beta \in \mathbf{S}$ *and* $\theta \in \Theta$ *such that:*

$$\begin{aligned} D(Tx, Ty) \leq & \beta(D(x,y))D(x,y) \\ & + \theta(D(y,Tx), D(x,Ty), D(x,Tx), D(y,Ty)), \end{aligned}$$

for all $x,\, y \in \mathcal{O}_T(\tilde{x})$*;*

iii) *The mapping defined by*

$$g\colon [0,\infty) \to [0,\infty), \quad g(t) = \beta(t)t,$$

is nondecreasing;

iv) $D(T^n\tilde{x}, T^n\tilde{x}) = 0$*, for all* $n \in \mathbb{N}$*;*

v) T *is* $\mathbf{A}$*-nondecreasing-continuous, and* $\mathbf{A}$*-nondecreasing.*

Then, the Picard sequence $\{T^n\tilde{x}\}$ *is D-convergent to a fixed point* ω *of* T*, and* $D(\omega, \omega) = 0$*.*

Furthermore, if there is another fixed point of T*,* ω'*, so that* $D(\omega, \omega') < \infty$*,* $D(\omega', \omega') < \infty$*, and the pair* (ω, ω') *checks the condition from* ii)*, then* $\omega = \omega'$*.*

The following example is meant to show the importance of our result.

Example 4. To begin with, let us consider the set $X = \left[0, \frac{1}{2}\right]$ endowed with the distance

$$D(x,y) = \begin{cases} \max\{x,y\}, & x \neq y \\ 0, & x = y. \end{cases}$$

One can see that (X, D) is a complete Jleli-Samet space, so we can define the following operator:

$$T\colon X \to X, \quad Tx = \ln(1+x) + \frac{x^2}{4}.$$

We can take now the Geraghty-type function

$$\beta\colon [0,\infty) \to (0,1), \quad \beta(x) = \begin{cases} \dfrac{\ln(1+x)}{x}, & \text{if } x \neq 0, \\ \\ 0, & \text{if } x = 0, \end{cases}$$

and $\theta(t,s,u,v) = tu$, $\theta \in \Theta$. T is a Geraghty-type contraction in the sense of our above theorems and has a unique fixed point, $x = 0$.

We emphasize that, for $x = \frac{1}{2}$ and $y = \frac{49}{100}$, the contractive inequality $D(Tx,Ty) \leq \beta(D(x,y))D(x,y)$ is no longer true, so T is not a Geraghty-contraction in the classic sense.

Extensions in the sense of Kannan and Ćirić can be formulated and proved as we have done in the previous chapter. It can be seen that Kannan contractive condition is strong enough and allows us to forget about hypothesis iii).

Theorem 15. *Let (X,D) be a Jleli-Samet space for which there is a preorder $\mathbf{A}$ such that the space is $\mathbf{A}$-nondecreasing-complete. Let $T\colon X \to X$ be a mapping and suppose that the following conditions are fulfilled:*

i) *There is $\tilde{x} \in X$, and $n_0 \in \mathbb{N}$ for which we have $\delta_{n_0}(D,T,\tilde{x}) < \infty$;*

ii) *There is a Geraghty-type function $\beta \in \mathbf{S}$ and $\theta \in \Theta$ such that:*

$$\begin{aligned} D(Tx,Ty) \leq& \beta\left(\frac{D(x,Tx)+D(y,Ty)}{2}\right)\frac{D(x,Tx)+D(y,Ty)}{2} \\ &+ \theta(D(y,Tx),D(x,Ty),D(x,Tx),D(y,Ty)), \end{aligned}$$

for all x, $y \in \mathcal{O}_T(\tilde{x})$;

iii) *$D(T^n\tilde{x},T^n\tilde{x}) = 0$, for all $n \in \mathbb{N}$;*

iv) *T is $\mathbf{A}$-nondecreasing-continuous, and $\mathbf{A}$-nondecreasing.*

Then, the Picard sequence $\{T^n\tilde{x}\}$ is D-convergent to a fixed point ω of T, and $D(\omega,\omega) = 0$.

Furthermore, if there is another fixed point of T, ω', with the properties $D(\omega,\omega') < \infty$, $D(\omega',\omega') = 0$, and the pair (ω,ω') fulfills the contractive condition from ii)*, then $\omega = \omega'$.*

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