National University of Science and Technology
POLITEHNICA Bucharest
Department of Mathematics & Informatics
Splaiul Independenţei 313, 060042 Bucharest, Romania

Geometric structures on manifolds. Submanifolds and applications

PhD student Crina Daniela NEACŞU
E-mail address: crina.neacsu@upb.ro

Abstract of the PhD Thesis

Supervised by Prof. Dr. habil. Gabriel-Eduard VÎLCU

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Contents

Abstract

Motivation and outlines. It is well known that the geometry of Riemannian structures, complex structures, contact structures, hypercomplex structures, quaternionic and Cauchy-Riemann structures belongs to the general theory of the G-structures, a domain of great interest in modern differential geometry, in global analysis and mathematical physics. On the other hand, submanifold theory is a vast and highly active research field, which has a key role in the current development of differential geometry. This field of study, which has many applications in other scientific branches, is still very promising, with many open problems worth investigating. The main goal of this PhD thesis is to obtain new results in this regard. The study will focus on four main directions: deducing optimal inequalities for curvature invariants of submanifolds, determining the fundamental properties of mixed 3-Sasaki and mixed 3-cosymplectic structures, as well as of statistical submersions between statistical manifolds of these types, establishing some criteria of stability/instability for T-space forms and the classification of production hypersurfaces with minimal isoquants. In the following, we will describe in detail the motivation of the problems addressed, the main results obtained, consequences of these results and some open problems deriving from them.

Obtaining simple relationships involving curvature invariants, both intrinsic and extrinsic in nature, is a key issue in the theory of Riemannian submanifolds [18]. A turning point in finding such relationships was the introduction of δ-invariants by Chen in [17]. In the work just referred to, the author introduced these curvature invariants (currently called Chen invariants) and for their usages he derived optimal inequalities involving the new invariants and the main extrinsic invariant (namely, squared mean curvature $\|H\|^2$) for submanifolds in real space forms. Since then, the theory of δ-invariants has become an active and fruitful area of research (for instance, see [18]).

On the other hand, the notion of Casorati curvature (which is an extrinsic invariant) was originally introduced in 1890 for surfaces in the Euclidean 3-space $\mathbb{E}^3$ (see [15]). The Casorati curvature gives us a better intuition about the curvature compared to Gaussian curvature. This is because the Gaussian curvature of a developable surface is

zero, whereas the surface is surely curved. Thus, from the viewpoint of our geometric intuition, the classical measure of curvature due to Gauss appears as not satisfactory. In view of this, Casorati proposed the notion of Casorati curvature of a surface $\mathcal{S}$ of $\mathbb{E}^3$ in the form $\mathcal{C} = \frac{1}{2}\left(\kappa_1^2 + \kappa_2^2\right)$, where k_1 and k_2 denote the principal curvatures of $\mathcal{S}$. Thus, it is clear that Casorati curvature $\mathcal{C}$ vanishes only at the planar points of $\mathcal{S}$, which really corresponds to our intuition on the measure of curvature. In the general case of a submanifold in a Riemannian manifold, the Casorati curvature is defined naturally as the normalized square length of the second fundamental form [24].

Over the past decade, various geometers have been working to obtain basic relationships of Chen type between the Casorati curvatures and the intrinsic invariants. See [21], where the main results obtained in this field have been excellently surveyed and some challenging research problems involving δ-Casorati curvatures were also proposed. Recall that the submanifolds attaining the equality case in Chen type inequalities are called ideal submanifolds and the term *ideal* is motivated from the fact that these submanifolds inherit the least possible tension from the ambient space. In the family of δ-invariants, we have the δ-Casorati curvatures which are defined using the Casorati curvature of the hyperplanes in the tangent space at a point.

In 2007, for an s-dimensional Riemannian submanifold in a Riemannian manifold, S. Decu, S. Haesen and L. Verstraelen introduced the normalized δ-Casorati curvatures $\hat{\delta}_C(s-1)$ and $\delta_C(s-1)$. They proved two sharp inequalities involving these new extrinsic invariants and the normalized scalar curvature ρ for submanifolds in real space forms [23]. One year later, the authors extended the notion to generalized normalized δ-Casorati curvatures $\hat{\delta}_C(r; s-1)$, for any real number $r > s(s-1)$, and $\delta_C(r; s-1)$, for any real number $0 < r < s(s-1)$ [24]. Following these generalized notions, they proved two sharp inequalities involving the new invariants and the scalar curvature. In connection to these two concepts of Casorati curvatures, namely normalized Casorati curvatures and generalized normalized Casorati curvatures, a lot of studies have been done. It is worth noting that $\hat{\delta}_C(s-1)$ and $\delta_C(s-1)$ are two different invariants, but in most of the studies we observe that they satisfy similar inequalities. The same situation happens with the pair of invariants $\hat{\delta}_C(r; s-1)$ and $\delta_C(r; s-1)$. In 2018, G.-E. Vîlcu derived an optimal inequality for Lagrangian submanifolds of complex space forms involving the normalized Casorati curvatures [63]. In the work we just mentioned above, a very unexpected result was obtained, namely the lower bounds of $\hat{\delta}_C(s-1)$ and $\delta_C(s-1)$ are different. To our knowledge, this is the first work of its kind in which the inequalities for $\hat{\delta}_C(s-1)$ and $\delta_C(s-1)$ are different. The equality cases of the inequalities are also discussed in the same paper, where the Casorati ideal submanifolds of complex space forms were classified. One year later, M. Aquib and other authors generalized the results of [63] to generalized normalized Casorati curvatures [4]. Moreover, the authors of [4] proposed two open problems concerning Legendrian submanifolds in Sasakian space forms and Lagrangian submanifolds in quaternionic space forms. Very recently, the first problem was completely solved in [38, 39]. In other words, the authors studied the contact version of [4] and [63]. The first issue addressed in this thesis consists in the study of the quaternionic counterpart of the problem, solving in particular the second question raised in [4].

The second problem tackled in the thesis is to investigate the generalized Casorati curvatures of statistical submanifolds in statistical manifolds. The notion of statistical manifold arises for the first time in Amari's paper [3] from the need of generalizing the concept of statistical model to statistical manifold. From now on, numerous and valuable studies that adapt the general theory of geometric structures on manifolds to statistical manifolds have been written (see, e.g., [27, 48, 58]). In the spirit of Casorati inequalities established in [24], Lee et. al proved in [40] some similar results in case of statistical manifolds, showing that the normalized scalar curvature is bounded by the Casorati curvatures of submanifolds in a statistical manifold of constant curvature. Our aim is to present an improvement of the inequalities established in [40], by considering a curvature tensor which is more natural in a statistical context, namely the statistical curvature tensor introduced by Opozda in [50]. This new tensor was introduced because it has all the symmetries that are necessary to a curvature tensor of type $(0, 4)$, unlike the classical Riemannian curvature tensor, which in a statistical setting no longer has all these symmetries [50].

The next problem addressed is the investigation of paraquaternionic structures, mixed 3-structures and Riemannian submersions in statistical setting. Recall that the concept of quaternionic structure of second kind, better known in the literature as paraquaternionic structure, was introduced by Libermann in [37]. The geometry of Riemannian manifolds equipped with paraquaternionic structures is an interesting research area and many remarkable results were established in the last decades, including applications in various fields like Lagrangian and Hamiltonian mechanical systems (see, e.g., [59, Chapter 6] and the references therein). On another hand, the notion of mixed 3-structure appeared in a natural way in the study of hypersurfaces of paraquaternionic manifolds performed by Ianuş, Mazzoco and Vîlcu in [35]. This concept was further exquisite by Caldarella and Pastore in [13], the authors introducing negative and positive mixed 3-

Sasakian structures, as well as negative and positive mixed 3-cosymplectic structures. Moreover, they proved that a positive (resp. negative) mixed 3-Sasakian space is an Einstein manifold with negative (resp. positive) Einstein constant. Several important results on the geometry of these manifolds and their submanifolds, as well as examples of such spaces, can be found in [12, 14, 36]. Motivated by all these studies, we are going to investigate the influence of the existence of quaternionic structures of second kind and mixed 3-structures on the geometry of statistical manifolds.

Another problem investigated in this thesis consists in the study of the stability of T-space forms. Recall that the identity map 1_M of a compact manifold M is one of the simplest examples of harmonic map. If 1_M is a stable harmonic map, then M is called stable and otherwise is said to be unstable. Harmonic maps are known to be geometric objects of great interest in quantum field theory and general relativity (see [53]), being intensively investigated in the last decades (for recent results see [56] and the references therein). In particular, the stability of harmonic maps is a problem of prime interest in mathematical physics and it was investigated by various authors (the reader is referred to the works [11, 32, 33, 51, 52] for several results on the stability of harmonic maps in almost complex, almost (para-)contact and almost quaternion geometry). Despite the simplicity of the identity function, the study of its stability is a difficult and intriguing subject as the second variation of this map is difficult to analyze. A first major result in this respect was obtained by Smith [55], who established a criterion for the stability of compact Einstein manifolds in terms of the first eigenvalue of the Laplace-Beltrami operator. Moreover, he proved that the identity map of any compact Kähler manifold is stable. Although most properties of Kählerian manifolds naturally extend to Sasakian manifolds, it is a surprising fact that most classical Sasaki spaces are unstable. In particular, the identity map of any odd-dimensional unit sphere is unstable [55]. Motivated by all these studies, we are interested in the extension of the stability results to the setting of T-manifolds of constant ϕ-sectional curvature. This is a natural problem, as T-manifolds stands as an umbrella for two important classes of manifolds, namely Kähler and cosymplectic manifolds. Moreover, according to Blair [9], T-manifolds along with K-manifolds and S-manifolds represent the three important classes of manifolds with structural groups $U(n) \times O(s)$.

Another topic addressed in this thesis is the study of hypersurfaces of the $(n+1)$-dimensional Euclidean space of high interest in production theory. It is well known that the production functions represent one of the most fundamental concepts used in economics [54]. Since production functions with n inputs can be naturally identified with hypersurfaces of the $(n+1)$-dimensional Euclidean space (see [60, 62]), the study of production models by means of differential geometric tools has become a fervent topic in recent years (see, e.g., [6, 61]). Therefore, we can find many geometric classification results for the basic production models utilised in the economic analysis, namely homogeneous [22], quasi-sum [7, 20], homothetic [19], quasi-homogeneous (or weighted-homogeneous) [2] and quasi-product production models [1, 28]. The proofs of these classification results are highly technical, generally involving a combination of techniques from mathematical analysis, differential geometry, ODE and PDE. Among geometric properties of production functions, those related to Gauss curvature and mean curvature of the corresponding production hypersurfaces are of primary interest. In particular, the minimality of quasi-sum production models was investigated by Y. Du, Y. Fu and X. Wang [26], the authors establishing the classification of minimal quasi-sum production hypersurfaces for dimensions 2 and 3. Recently, Y. Luo and X. Wang [42] extended the classification of minimal hypersurfaces to the case of quasi-product production models. Recall that a hypersurface $\mathcal{H}$ is said to be minimal if the mean curvature of $\mathcal{H}$ vanishes identically. The minimality is a fundamental property originally investigated in differential geometry (see, e.g., [25]), but due to its significance and potential applications, was subsequently considered in many other fields, such as calculus of variations, potential theory, complex analysis, architecture, general relativity, molecular engineering, sculpture, aviation, economics and materials science. We focuss our study on the minimality of isoquants, a key concept in both production and supply theory, which was introduced independently by A.L. Bowlley, R. Frisch, C.W. Cobb and A.P. Lerner (for details on the paternity of the notion, see [41]). Roughly speaking, an isoquant geometrically illustrates all combination of inputs used in a production process that assure a certain level of output. Despite the importance of the geometry of isoquants in making the best decision for cost minimization and therefore for profit maximization, no classification results have been obtained for production models with minimal isoquants. This is due to the lack of appropriate methods for solving the nonlinear PDE that characterizes the property of minimality in this setting. Our goal is to show that we can obtain such a classification result in the case of quasi-product type production models.

Thesis description: structure and content.

Lagrangian submanifolds, a class of Riemannian submanifolds that arose naturally in the context of Hamiltonian mechanics, play an important role in some modern theories of physics. In Chapter 1, **Lagrangian submanifolds in quaternionic Kähler manifolds**, whose content is based on paper [5], using an optimization technique on submanifolds immersed in Riemannian manifolds, we first obtain some sharp inequalities for δ-Casorati curvature invariants of Lagrangian submanifolds in quaternionic space forms, i.e. quaternionic Kähler manifolds of constant q-sectional curvature.

Theorem 1. [5] *If M^s is a Lagrangian submanifold of a quaternionic space form $\hat{M}^{4s}(c)$, then we have the following.*

(i) For any real number r, such that $0 < r < s(s-1)$:

$$\rho \leq \frac{\delta_C(r; s-1)}{s(s-1)} + \frac{c}{4} - \frac{2r^2}{(s-1)[s^2 + s(r-1) + r]} \|H\|^2. \tag{1}$$

(ii) For any real number r, such that $r > s(s-1)$:

$$\rho \leq \frac{\widehat{\delta}_C(r; s-1)}{s(s-1)} + \frac{c}{4} - \frac{2s[s^2 + s(r-1) - 2r]}{(s-1)[s^2 + s(r-1) + r]} \|H\|^2. \tag{2}$$

A Casorati ideal submanifold in a Riemannian manifold is one that satisfies equality pointwise in an optimal inequality involving δ-Casorati curvature invariants. We show that in the class of Lagrangian submanifolds in quaternionic space forms, there are only two subclasses of ideal Casorati submanifolds, namely the family of totally geodesic submanifolds and a particular subfamily of H-umbilical submanifolds. Recall that, motivated by the fact that there are no non-trivial totally umbilical Lagrangian submanifolds in quaternionic space forms, the concept of Lagrangian H-umbilical submanifold of a quaternionic space form was introduced in [49].

Definition 1. [49] *A non-totally geodesic Lagrangian submanifold M^s of a quaternionic space form $\hat{M}^{4s}(c)$ is called a Lagrangian H-umbilical submanifold if its second fundamental form satisfies*

$$\begin{cases} \sigma(e_1, e_1) = \sum_{\varsigma} \lambda_\varsigma J_\varsigma e_1, \\ \sigma(e_2, e_2) = \cdots = \sigma(e_s, e_s) = \sum_{\varsigma} \mu_\varsigma J_\varsigma e_1, \\ \sigma(e_1, e_j) = \sum_{\varsigma} \mu_\varsigma J_\varsigma e_j, \quad j = 2, \ldots, s, \\ \sigma(e_i, e_j) = 0, \quad i \neq j, \quad i, j = 2, \ldots, s, \end{cases} \tag{3}$$

for some functions $\{\lambda_\varsigma, \mu_\varsigma\}_{\varsigma=1,2,3}$ with respect to some orthonormal local frame fields.

We obtained the following results.

Theorem 2. [5] *Suppose M is a Lagrangian submanifold of a quaternionic space form $\hat{M}^{4s}(c)$. Then M is an ideal submanifold for* (2) *if and only if M is totally geodesic.*

Theorem 3. [5] *Suppose M is a Lagrangian Casorati ideal submanifold for* (1). *Then M is either a totally geodesic Lagrangian submanifold, or an H-umbilical submanifold satisfying* (3) *with* $\mu_\varsigma = \frac{s^2-s+2r}{r}\lambda_\varsigma$, $\varsigma = 1, 2, 3$.

At the end of this chapter, we provided some examples to illustrate the obtained results. In particular, we point out that an entire family of ideal Casorati Lagrangian submanifolds can be constructed using the concept of quaternionic extensor introduced by Oh and Kang in the early 2000s (see [49]).

Example 1. [5] The real projective space $\mathbb{R}P^n\left(\frac{c}{4}\right)$ (with constant sectional curvature $\frac{c}{4}$) can be immersed naturally in $\mathbb{H}P^n(c)$ as a totally geodesic Lagrangian submanifold (see, e.g., [29, 43]). This immersion is defined in terms of homogeneous coordinates as $\iota : \mathbb{R}P^n\left(\frac{c}{4}\right) \to \mathbb{H}P^n(c)$, $\iota(u) = (u, 0, 0, 0)$. Hence ι provides us one of the simplest examples of ideal Casorati Lagrangian submanifold for (1) and (2).

Example 2. [5] The Riemannian n-dimensional sphere provides us a second remarkable example of totally geodesic submanifold of $\mathbb{H}P^n(c)$ (see [16]). Obviously, this is an ideal Casorati Lagrangian submanifold for (1) and (2).

Example 3. [5] An entire family of ideal Casorati Lagrangian submanifolds for (1) and (2) can be obtained through the quaternionic extensors introduced in [49]. Recall that a quaternionic extensor of an isometric immersion $G : M^{n-1} \to \mathbb{E}^m$ of a Riemannian manifold of dimension $n-1$ into the Euclidean m-dimensional space $\mathbb{E}^m$, is the tensorial product immersion of a unit speed curve $F : I \to \mathbb{H}$ in the quaternionic plane, and the immersion G, denoted by $\phi = F \otimes G : I \times M^{n-1} \to \mathbb{H} \otimes \mathbb{E}^m = \mathbb{H}^m$ and defined as $\phi(t, p) = F(t) \otimes G(p),\ \forall\, t \in I,\ p \in M^{n-1}$.

Applying [49, Proposition 3.3], it follows that considering the inclusion $\iota : S^{n-1} \to \mathbb{E}^n$ of the unit hypersphere S^{n-1} of $\mathbb{E}^n$, and the unit speed curve F in $\mathbb{H}$ defined by $F(t) = (t + a)q$, for some $a \in \mathbb{R}$ and some unitary quaternion $q \in \mathbb{H}$, we derive that the quaternionic extensor $\phi = F \otimes \iota$ defines a totally geodesic Lagrangian submanifold of $\mathbb{H}^n$. Hence $\phi = F \otimes \iota$ provides examples of ideal Casorati Lagrangian submanifolds of the quaternionic Euclidean space $\mathbb{H}^n$ for (1) and (2) for all $a \in \mathbb{R}$ and unit quaternion $q \in \mathbb{H}$. Note that if the generating curve F is any unit speed curve in $\mathbb{H}$ different from $F(t) = (t + a)q$, for some $a \in \mathbb{R}$ and some unitary quaternion $q \in \mathbb{H}$, then every quaternionic extensor $\phi = F \otimes \iota$ defines a Lagrangian submanifold of $\mathbb{H}^n$, with the second fundamental form σ satisfying (3) for $\lambda_\varsigma =< F'', J_\varsigma F' >$ and $\mu_\varsigma =< \left(\frac{F}{||F||}\right)', J_\varsigma\left(\frac{F}{||F||}\right) >$, where $\varsigma = 1, 2, 3$. Hence, in this case, $\phi = F \otimes \iota$ defines a Lagrangian H-umbilical submanifold of $\mathbb{H}^n$.

In [40], Lee et al. proved some optimal inequalities involving the normalized scalar curvature and normalized δ-Casorati curvatures of submanifolds in a statistical manifold of constant curvature, later generalized by Bansal et al in [8] to the case of generalized normalized δ-Casorati curvatures. In Chapter 2, **Curvature inequalities for statistical submanifolds of statistical manifolds**, whose content is based on the article [44], we will improve all these inequalities, by considering a more natural curvature tensor field in statistical setting, called the statistical curvature tensor, originally introduced by Opozda in [50]. Recall that a statistical manifold $(\overline{M}, \bar{g}, \overline{\nabla})$ is a Riemannian manifold endowed with a pair of torsion free affine connections $\bar{\nabla}$ and $\bar{\nabla}^*$ such that

$$Z\bar{g}(X,Y) = \bar{g}(\overline{\nabla}_Z X, Y) + \bar{g}(X, \overline{\nabla}^*_Z Y), \forall X, Y, Z \in \Gamma(T\overline{M}).$$

The connections $\overline{\nabla}$ and $\overline{\nabla}^*$ are called *dual (conjugate) connections.* In terms of the Levi-Civita connection $\overline{\nabla}^{\circ}$ of the metric $\bar{g}$, the affine connection $\overline{\nabla}$ has always a dual connection $\overline{\nabla}^*$ satisfying $\overline{\nabla} + \overline{\nabla}^* = 2\overline{\nabla}^{\circ}$. Let $\overline{R}$ and $\overline{R}^*$ be the curvature tensor fields of $\overline{\nabla}$, respectively $\overline{\nabla}^*$. A statistical structure $(\overline{\nabla}, g)$ is said to be of *constant curvature* $c \in \mathbb{R}$ if

$$\overline{R}(X,Y)Z = c\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\}, \forall X, Y, Z \in \Gamma(T\overline{M}).$$

With the notations above, we define S be *the statistical curvature tensor* as [50]:

$$S(X,Y)Z = \frac{1}{2}\{R(X,Y)Z + R^*(X,Y)Z\}.$$

A statistical manifold $(\overline{M}, \bar{g}, \overline{\nabla})$ is said to be a statistical space form if S has the following expression for all vector fields X, Y, Z on $\overline{M}$:

$$S(X,Y)Z = c\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\},$$

where c is a real constant. Such a space is denoted by $\overline{M}(c)$.

For a statistical submanifold M of a statistical manifold $(\overline{M}, \bar{g}, \overline{\nabla})$, we denote by h and h^* the imbedding curvature tensors with respect to the dual connections $\overline{\nabla}$ and $\overline{\nabla}^*$ and by H and H^* be the corresponding mean curvature vector fields. It is easy to see that $2H^{\circ} = H + H^*$, where H° denotes the mean curvature field of M defined through the second fundamental form h° with respect to the Levi-Civita connection ∇° on M. If $h^{\circ} = 0$, then M is called totally geodesic (with respect to the Levi-Civita connection ∇°).

We established the next improvement of the inequalities stated in [40] and [8].

Theorem 4. [44] *Let M^n be a statistical submanifold of a statistical space form $\overline{M}^m(c)$. Then:*

i. The normalized δ-Casorati curvatures $\delta_C(r, n-1)$ *and* $\delta_C^*(r, n-1)$ *satisfy*

$$\rho \leq \frac{2\delta_C^\circ(r, n-1)}{n(n-1)} + \frac{C^0}{2(n-1)} - \frac{4n}{n-1}\left\|H^\circ\right\|^2 + \frac{2n}{n-1}g(H, H^*) + c, \tag{4}$$

where $2\delta_C^0(r, n-1) = \delta_C(r, n-1) + \delta_C^*(r, n-1)$ *and* $2C^0 = \mathcal{C} + \mathcal{C}^*$.

ii. The normalized δ-Casorati curvatures $\hat{\delta}_C(r, n-1)$ *and* $\hat{\delta}_C^*(r, n-1)$ *satisfy*

$$\rho \leq \frac{2\hat{\delta}_C^\circ(r, n-1)}{n(n-1)} + \frac{C^0}{2(n-1)} - \frac{4n}{n-1}\left\|H^\circ\right\|^2 + \frac{2n}{n-1}g(H, H^*) + c, \tag{5}$$

where $2\hat{\delta}_C^0(r, n-1) = \hat{\delta}_C(r, n-1) + \hat{\delta}_C^*(r, n-1)$.

As an immediate consequence of the above theorem, we deduce the following result.

Corollary 1. [44] *Let* M^n *be a statistical submanifold of a statistical space form* $\overline{M}^m(c)$. *Then:*

i. The normalized δ-Casorati curvatures $\delta_C(n-1)$ *and* $\delta_C^*(n-1)$ *satisfy:*

$$\rho \leq 2\delta_C^\circ(n-1) + \frac{C^0}{2(n-1)} - \frac{4n}{n-1}\left\|H^\circ\right\|^2 + \frac{2n}{n-1}g(H, H^*) + c. \tag{6}$$

where $2\delta_C^0(n-1) = \delta_C(n-1) + \delta_C^*(n-1)$.

ii. The normalized δ-Casorati curvatures $\hat{\delta}_C(n-1)$ *and* $\hat{\delta}_C^*(n-1)$ *satisfy:*

$$\rho \leq 2\hat{\delta}_C^\circ(n-1) + \frac{C^0}{2(n-1)} - \frac{4n}{n-1}\left\|H^\circ\right\|^2 + \frac{2n}{n-1}g(H, H^*) + c. \tag{7}$$

where $2\hat{\delta}_C^0(n-1) = \hat{\delta}_C(n-1) + \hat{\delta}_C^*(n-1)$.

Investigating the equality cases of the above inequalities, we established the next results.

Theorem 5. [44] *Let* M^n *be a statistical submanifold of a statistical space form* $\overline{M}^m(c)$. *Then the case of equality of any of the inequalities* (4) *and* (5) *occurs at a point* $x \in M$ *if and only if the imbedding curvature tensors* h *and* h^* *are related at* x *by* $h^* = -h$.

Corollary 2. [44] *Let* M^n *be a statistical submanifold of a statistical space form* $\overline{M}^m(c)$. *Then the case of equality of any of the inequalities* (4) *and* (5) *occurs identically at any point* $x \in M$ *if and only if* M *is a totally geodesic submanifold of* $\overline{M}^m(c)$ *with respect to the Levi-Civita connection.*

In order to illustrate the results, we constructed an example, as follows.

Example 4. [44] Consider $\mathbb{H}^{m+1} = \{(x_1, ..., x_{m+1}) \in \mathbb{R}^{m+1} | x_{m+1} > 0\}$ equipped with the natural metric $\bar{g} = \frac{1}{(x_{m+1})^2} \sum_{i=1}^{m+1} (dx_i)^2$. We take the affine connection $\bar{\nabla}$ given by

$$\bar{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0,\ 1 \leq i \neq j \leq m,\ \bar{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_{m+1}} = \bar{\nabla}_{\frac{\partial}{\partial x_{m+1}}} \frac{\partial}{\partial x_i} = 0,\ 1 \leq i \leq m,$$

$$\bar{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_i} = \frac{2}{x_{m+1}} \frac{\partial}{\partial x_{m+1}},\ 1 \leq i \leq m,\ \bar{\nabla}_{\frac{\partial}{\partial x_{m+1}}} \frac{\partial}{\partial x_{m+1}} = \frac{1}{x_{m+1}} \frac{\partial}{\partial x_{m+1}}.$$

It is known that $(\bar{\nabla}, \bar{g})$ is a statistical structure of constant curvature 0 on $\mathbb{H}^{m+1}$ (see [30]). It follows that $(\bar{\nabla}^*, \bar{g})$ is also a statistical structure of constant curvature 0 on $\mathbb{H}^{m+1}$. Therefore we conclude that $\mathbb{H}^{m+1}$ is a statistical space form of constant curvature 0. We consider now an immersion $\iota : \mathbb{H}^{n+1} \to \mathbb{H}^{m+1}$ defined by $\iota(x_1, ..., x_{n+1}) = (0, \ldots, 0, x_1, ..., x_{n+1})$, where $n < m$. Then it is easy to see that ι is a totally geodesic immersion providing a natural example of statistical submanifold satisfying the equality cases of all inequalities stated above, namely (4), (5), (6) and (7).

In Chapter 3, **Mixed 3-Sasakian statistical manifolds and statistical submersions**, whose content is based on paper [45], we investigate the influence of the existence of paraquaternionic and mixed 3-structures on the geometry of statistical manifolds. Recall that the notion of quaternionic structure of second kind, better known in the literature as paraquaternionic structure, was introduced by Libermann in [37]. The basic properties of manifolds equipped with such kind of structures were established in [31]. Mixed 3-structures are known to be the counterpart in odd dimension of the quaternionic structures of second kind. The concept of mixed 3-structure arose naturally in the study of lightlike hypersurfaces of paraquaternionic manifolds [35].

We first derived the main properties of statistical manifolds equipped with such kind of structures, focusing on the case of mixed 3-Sasakian statistical manifolds and paraquaternionic Kähler-like statistical manifolds.

Theorem 6. [45] *If $(M, \nabla, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ is a mixed 3-structure statistical manifold, then we have for all vector fields X, Y and Z on M:*

$$(\nabla_X \omega_\alpha)(Y, Z) = g(Y, \nabla^*_X \varphi_\alpha Z - \varphi_\alpha \nabla_X Z) \tag{8}$$

and

$$\nabla_X \varphi_\alpha Y - \varphi_\alpha \nabla^*_X Y = (\nabla^\circ_X \varphi_\alpha) Y + K_X \varphi_\alpha Y + \varphi_\alpha K_X Y, \tag{9}$$

for $\alpha = 1, 2, 3$, where $\omega_\alpha(Y, Z) = g(Y, \varphi_\alpha Z)$ and $K_X Y = \nabla_X Y - \nabla^\circ_X Y$.

Theorem 7. [45] *(i) A mixed 3-structure statistical manifold* $(M, \nabla, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ *is mixed 3-Sasakian if and only if:*

$$\nabla_X \varphi_\alpha Y - \varphi_\alpha \nabla^*_X Y = \tau_\alpha[g(X,Y)\xi_\alpha - \varepsilon_\alpha \eta_\alpha(Y)X] \tag{10}$$

and

$$\nabla_X \xi_\alpha = \varepsilon_\alpha[g(\nabla_X \xi_\alpha, \xi_\alpha)\xi_\alpha - \varphi_\alpha X] \tag{11}$$

or, equivalently, if and only if

$$\nabla^*_X \varphi_\alpha Y - \varphi_\alpha \nabla_X Y = \tau_\alpha[g(X,Y)\xi_\alpha - \varepsilon_\alpha \eta_\alpha(Y)X] \tag{12}$$

and

$$\nabla^*_X \xi_\alpha = \varepsilon_\alpha[g(\nabla^*_X \xi_\alpha, \xi_\alpha]\xi_\alpha - \varphi_\alpha X), \tag{13}$$

for all vector fields X *and* Y *on* M.

(ii) A mixed 3-structure statistical manifold $(M, \nabla, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ *is mixed 3-cosymplectic if and only if:*

$$\nabla_X \varphi_\alpha Y = \varphi_\alpha \nabla^*_X Y \tag{14}$$

or, equivalently,

$$\nabla^*_X \varphi_\alpha Y = \varphi_\alpha \nabla_X Y \tag{15}$$

for all vector fields X *and* Y *on* M.

Theorem 8. [45] *Suppose* (M, ∇, σ, g) *is an almost Hermite-like paraquaternionic statistical manifold. Then* (M, ∇, σ, g) *is a paraquaternionic Kähler-like statistical manifold if and only if* $(M, \nabla^*, \sigma^*, g)$ *is a paraquaternionic Kähler-like statistical manifold.*

Theorem 9. [45] *Suppose* (M, ∇, σ, g) *is an almost Hermite-like paraquaternionic statistical manifold. Then* (M, ∇, σ, g) *is a parahyper-Kähler-like statistical manifold, if and only if so is* $(M, \nabla^*, \sigma^*, g)$.

Example 5. [45] Let (M, ∇, P, g) be an almost para-Hermitian-like manifold (for examples of such manifolds see [64, Section 5]). Next, we prove that the tangent bundle TM of M can be endowed with an almost Hermite-like paraquaternionic statistical structure. First, we note that TM can be equipped with a Sasaki-type metric G defined by: $G(A,B) = g(kA, kB) + g(\pi_* A, \pi_* B)$, for all vector fields A, B on TM, where $\pi : TM \to M$ is the natural projection and k is the connection map associated with the Levi-Civita connection of the metric g. We would like to remark at this stage that if $X \in \Gamma(TM)$, then there is a unique vector field on TM, denoted by X^h and called the horizontal lift of

X, and a unique vector field on TM denoted by X^v and called the vertical lift of X, such that for all $U \in TM$ we have: $\pi_* X^h_U = X_{\pi(U)},\ \pi_* X^v_U = 0_{\pi(U)},\ kX^h_U = 0_{\pi(U)},\ kX^v_U = X_{\pi(U)}$. We recall now that, according to Theorem 3.1 in [34], one can define a torsion free linear connection ∇' on TM compatible to G. Hence, it follows that (TM, ∇', G) is a statistical manifold. Using now the almost product structure P on M, we can also define three tensor fields $\{J_1, J_2, J_3\}$, on TM as follows:

$$\begin{cases} J_3 X^h = X^v \\ J_3 X^v = -X^h \end{cases}, \quad \begin{cases} J_2 X^h = (\varphi X)^v \\ J_2 X^v = (\varphi X)^h \end{cases}, \quad \begin{cases} J_1 X^h = -(\varphi X)^h \\ J_1 X^v = (\varphi X)^v \end{cases}.$$

By a direct computation we obtain that $H = (\{J_1, J_2, J_3\})$ is an almost para-hypercomplex structure on TM. Moreover, considering the 3-subbundle σ generated by H, it follows immediately that (TM, ∇', σ, G) is an almost Hermite-like paraquaternionic statistical manifold. In addition, we also obtain that the quadruplet (TM, ∇', σ, G) is a parahyper-Kähler-like statistical manifold, provided that (M, ∇, P, g) is a flat para-Kähler-like statistical manifold.

Next part of this chapter investigates statistical submersions with total space a mixed 3-Sasakian manifold or a paraquaternionic Kähler-like statistical manifold.

Definition 2. [58] *If (M, ∇, g) and (N, ∇^N, g_N) are statistical manifolds, then a semi-Riemannian submersion $\pi : M \to N$ is called a* statistical submersion *if $\pi_*(\nabla_X Y)_p = (\nabla^N{}_{X'} Y')_{\pi(p)}$ for all basic vector fields X, Y on M which are π-related to X' and Y' on N, and $p \in M$.*

Recall that the vectors tangent to fibres of a statistical submersion $\pi : M \to N$ are called *vertical* and those normal to fibers are called *horizontal*. We denote by $\mathcal{V}$ the vertical distribution, by $\mathcal{H}$ the horizontal distribution, and by v and h the vertical and horizontal projections. We also recall that an horizontal vector field X on M is called *basic*, if X is π-related to a vector field X' on N. It is clear that every vector field X' on N has a unique horizontal lift X on M and X is basic.

If $\pi : M \to N$ is a statistical submersion, then it is known that any fiber admits a canonical statistical structure [58]. Hence any fiber is a statistical manifold. The affine connections induced on fibres by the dual connections ∇ and ∇^* from M are denoted by $\widehat{\nabla}$ and $\widehat{\nabla}^*$, respectively. Note that $\widehat{\nabla}$ and $\widehat{\nabla}^*$ are torsion free and conjugate to each other with respect to the induced metric on fibres.

For a statistical submersion $\pi : M \to N$ it is also possible to define as in the case of semi-Riemannian submersions, the O'Neill tensor fields T and A by

$$T(E, F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF,\ A(E, F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF,$$

for $E, F \in \Gamma(TM)$. In a similar way, we can consider the O'Neill tensor fields T^* and A^* on M by putting ∇^* instead of ∇ in the above equation. If $T_U V = 0$, for all vertical vector fields U and V on M, then π is said to have *isometric fibers* [57].

Definition 3. [45] *Let $(M, \nabla, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ be a mixed 3-structure statistical manifold and (N, ∇^N, g_N) be a statistical manifold. Then a statistical submersion $\pi : M \to N$ is called*

(i) invariant *if $\varphi_\alpha(Ker\pi_*) \subset Ker\pi_*$, for $\alpha \in \{1, 2, 3\}$.*

(ii) anti-invariant *if $\varphi_\alpha(Ker\pi_*) \subset (Ker\pi_*)^\perp$, for $\alpha \in \{1, 2, 3\}$.*

Theorem 10. [45] *Let $(M, \nabla, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ be a mixed 3-structure statistical manifold and (N, ∇^N, g_N) be a statistical manifold. If $\pi : M \to N$ is an anti-invariant statistical submersion, then the structure vector fields ξ_1, ξ_2 and ξ_3 can not be all vertical.*

Theorem 11. [45] *Let $(M, \nabla, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ be a mixed 3-Sasakian statistical manifold and (N, ∇^N, g_N) be a statistical manifold. If $\pi : M \to N$ is a statistical submersion such that the structure vector fields ξ_1, ξ_2 and ξ_3 are all horizontal, then π is anti-invariant.*

Theorem 12. [45] *Let $(M, \nabla, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ be a mixed 3-structure statistical manifold and (N, ∇^N, g_N) be a statistical manifold. If $\pi : M \to N$ is an invariant statistical submersion, then the structure vector fields ξ_1, ξ_2 and ξ_3 are either vertical or horizontal.*

Theorem 13. [45] *Let $(M, \nabla, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ be a mixed 3-Sasakian statistical manifold (resp. a mixed 3-cosymplectic statistical manifold) and (N, ∇^N, g_N) be a statistical manifold. If $\pi : M \to N$ is an invariant statistical submersion such that the structure vector fields ξ_1, ξ_2 and ξ_3 are vertical, then any fiber of the submersion is a mixed 3-Sasakian statistical manifold (resp. a mixed 3-cosymplectic statistical manifold).*

Theorem 14. [45] *Let $(M, \nabla, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ be a mixed 3-Sasakian statistical manifold or a mixed 3-cosymplectic statistical manifold and (N, ∇^N, g_N) be a statistical manifold. If $\pi : M \to N$ is an invariant statistical submersion such that the structure vector fields ξ_1, ξ_2 and ξ_3 are vertical, then π is a statistical submersion with isometric fibers.*

Theorem 15. [45] *Suppose $\pi : M \to M'$ is a paraquaternionic Kähler-like statistical submersion. Then:*

(i) The fibres and the base of π are paraquaternionic Kähler-like statistical manifolds.

(ii) π is a statistical submersion with isometric fibers.

As an immediate consequence of the above theorem, we have the next result.

Corollary 3. [45] *Suppose $\pi : M \to M'$ is a parahyper-Kähler-like statistical submersion. Then the fibres and the base of π are parahyper-Kähler-like statistical manifolds. Moreover, π has isometric fibers.*

This chapter ends with an illustrative example.

Example 6. [45] If (M, ∇, σ, g) is an almost Hermite-like paraquaternionic statistical manifold, then it is known that the tangent bundle TM equipped with the Sasaki metric G can be endowed with a linear connection ∇' such that (TM, ∇', G) is a statistical manifold (see [34]). Moreover, we define for any canonical local basis $\{J_1, J_2, J_3\}$ of σ the next three tensor fields on TM: $J'_\alpha X^h = (J_\alpha X)^h$, $J'_\alpha X^v = (J_\alpha X)^v$, $\alpha \in \{1, 2, 3\}$. If $\sigma' = Span\{J'_1, J'_2, J'_3\}$, then it is easy check that $(TM, \nabla', \sigma', g')$ is an almost Hermite-like paraquaternionic statistical manifold and the natural projection $\pi : TM \to M$ provide us a non-trivial example of almost Hermite-like paraquaternionic statistical submersion. More than that, π is a parahyper-Kähler-like statistical submersion with isometric fibers, provided that (M, ∇, σ, g) is flat.

In Chapter 4, **Stability of T-space forms**, whose content is based on paper [46], we investigate the stability of compact T-manifolds of constant ϕ-sectional curvature, i.e. compact T-space forms. Despite the simplicity of the identity function, the study of its stability is a difficult and intriguing subject as the second variation of this map is difficult to analyze and generally requires an interplay of techniques (see, e.g., [32, 55]).

In the first main result of this chapter, using the second variation formula for the identity function 1_M of a compact T-space form M, we find the condition under which this function is stable. Thus, we established the next result.

Theorem 16. [46] *Any compact T-space form M of constant φ-sectional curvature $c \geq 0$ is stable.*

From Theorem 16, we can deduce as corollaries that the identity map of any complex space form $M(c)$ is stable, as well the identity map of any cosymplectic space form $M(c)$ is stable, provided that $c \geq 0$. We would like to emphasize that these results are not as general as those known in the literature for Kähler and cosymplectic manifolds (these manifolds being known to be stable under compactness assumptions - see [10, 11]), but they show us that T-manifolds represent a much more interesting family from the point of view of the geometry of the identity map, as there is a possibility that these manifolds are unstable.

In the second main result of this chpater, making use of the Weitzenböck formula, we prove that a $(2n+s)$-dimensional compact T-space form of constant φ-sectional curvature $c \leq 0$ is unstable, provided that the first eigenvalue of the Laplace-Beltrami operator has an upper bound in terms of n and c, as follows.

Theorem 17. [46] *Let M be a $(2n+s)$-dimensional T-space form of constant φ-sectional curvature $c \leq 0$. If the first eigenvalue λ_1 of the Laplace-Beltrami operator satisfies $\lambda_1 < -c(n+1)$, then 1_M is unstable.*

In Chapter 5, **On the minimality of isoquants of production hypersurfaces**, whose content is based on [47] we investigate the geometry of isoquants of a production function f with arbitrary number of inputs n, this being a key concept used in making the best decision for optimizing the production costs. Recall that an *isoquant* represents the set of all possible combinations of inputs $(x_1, ..., x_n)$ used in a production process that produce a specified level of output f^*. Such an isoquant is denoted by $\mathcal{H}_{f^*}$ and it is also called as *the set of level* f^* of the production function f. It is known that any production function f with n inputs can be identified with a nonparametric hypersurface $\mathcal{H}$ of the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$, defined by [62]

$$\mathcal{H} = \{(x_1, ..., x_n, f(x_1, ..., x_n)) \mid (x_1, \ldots, x_n) \in D\}. \tag{16}$$

Through this identification, the basic properties of production models can be reinterpreted in terms of the geometry of their graphs [19]. This hypersurface is called the *production hypersurface* [62].

The study undertaken in this chapter is focussed on the isoquants of the hypersurfaces corresponding to the quasi-product production models with n inputs. Recall that a production function f is called *quasi-product* if it is expressed as [1]

$$f(x_1, \ldots, x_n) = F\left(\prod_{i=1}^{n} g_i(x_i)\right), \tag{17}$$

where g_i $(i = 1, \ldots, n)$ and F are positive and strict monotone continuous functions. In the particular case where F is the identity function, the production function f given above is said to be a *product production function* or a *factorable production function.* As important particular examples of such production models, we have the famous Cobb-Douglas, CES, transcendental and Mitscherlich-Spillman production models (see [1, 28, 42] for details).

In the main result of this chapter, we give the complete classification of quasi-product production functions whose isoquants are minimal, showing that there are exactly 9 production models with such a property, as follows.

Theorem 18. [47] *Suppose $D \subset \mathbb{R}^n$ is an open domain and let $f : D \to \mathbb{R}_+$ be a quasi-product production function given by (17), where $F, g_1, \dots, g_n$ are twice differentiable functions. Then f has minimal isoquants iff, modulo a suitable translation, f reduces to the one of the next functions:*

(a)

$$f(x_1, \dots, x_n) = \alpha \prod_{i=1}^{n-1} (\beta_i x_i + \gamma_i) g_n(x_n) + \beta,$$

where $\alpha, \beta, \beta_i, \gamma_i$ are real constants with $\alpha \neq 0$ and $\beta_i \neq 0$ for $i = 1, \dots, n$, such that $f > 0$ on D.

(b)

$$\begin{aligned} f(x_1, \dots, x_n) &= \alpha \prod_{i=1}^{j} \left(C_{1i} e^{x_i \sqrt{A_i}} + C_{2i} e^{-x_i \sqrt{A_i}} \right) \\ &\times \prod_{i=j+1}^{n-1} \left[C_{1i} \cos\left(x_i \sqrt{-A_i} \right) + C_{2i} \sin\left(x_i \sqrt{-A_i} \right) \right] \\ &\times g_n(x_n) + \beta, \end{aligned}$$

where j is any number from the set $\{1, \dots, n-2\}$, while $\alpha, \beta, A_i, C_{1i}, C_{2i}$ are real constants with $A_1, \dots, A_j > 0$, $A_{j+1}, \dots, A_{n-1} < 0$ and $\sum_{i=1}^{n-1} A_i = 0$, such that $f > 0$ on D.

(c)

$$\begin{aligned} f(x_1, \dots, x_n) &= \alpha \prod_{i=1}^{r} (\beta_i x_i + \gamma_i) \prod_{i=r+1}^{s} \left(C_{1i} e^{x_i \sqrt{A_i}} + C_{2i} e^{-x_i \sqrt{A_i}} \right) \\ &\times \prod_{i=s+1}^{n-1} \left[C_{1i} \cos\left(x_i \sqrt{-A_i} \right) + C_{2i} \sin\left(x_i \sqrt{-A_i} \right) \right] \\ &\times g_n(x_n) + \beta, \end{aligned}$$

where r is any number from the set $\{1, \dots, n-3\}$, s is any number from the set $\{r+1, \dots, n-2\}$, while $\alpha, \beta, \beta_i, \gamma_i, A_i, C_{1i}, C_{2i}$ are real constants with $A_{r+1}, \dots, A_s > 0$, $A_{s+1}, \dots, A_{n-1} < 0$ and $\sum_{i=r+1}^{n-1} A_i = 0$, such that $f > 0$ on D.

(d)

$$f(x_1, \dots, x_n) = b \ln \left[A \prod_{i=1}^{n-1} e^{A x_i^2/2 + \alpha_i x_i} g_n(x_n) \right] + a,$$

where a, b, A, α_i are real constants with $A \neq 0$, $b \neq 0$ and $\sum_{i=1}^{n-1} A_i = 0$, such that f is well defined and strictly positive on D.

(e)

$$f(x_1,\dots,x_n)=\frac{A}{\alpha+1}\prod_{i=1}^{n-1}\left[(1+\alpha)x_i-\alpha_i\right]\times[g_n(x_n)]^{\alpha+1}+a,$$

where A,α,a,α_i *are real constants with* $\alpha\neq 0$, $\alpha\neq -1$ *and* $A\neq 0$ *such that* $f>0$ *on* D.

(f)

$$\begin{aligned}f(x_1,\dots,x_n) &= \frac{A}{\alpha+1}\prod_{i=1}^{j}\left[\left(\alpha_i^{2\sqrt{\frac{A_i}{1+\alpha}}}e^{-2\sqrt{A_i(1+\alpha)}x_i}-1\right)e^{\sqrt{A_i(1+\alpha)}x_i}\right]\\ &\quad\times\prod_{i=j+1}^{n-1}\cos\left(\sqrt{-\frac{A_i}{1+\alpha}}[-(1+\alpha)x_i+\beta_i]\right)\times[g_n(x_n)]^{\alpha+1}+a,\end{aligned}$$

where A,α,a,β_i,A_i *are real constants with* $\alpha\neq 0$, $\alpha>-1$, $A\neq 0$, $A_i>0$ *for* $i=1,\dots,j$, $A_i<0$ *for* $i=j+1,\dots,n-1$ *and* $\sum_{i=1}^{n-1}A_i=0$, *such that* $f>0$ *on* D.

(g)

$$\begin{aligned}f(x_1,\dots,x_n) &= \frac{A}{\alpha+1}\prod_{i=1}^{j}\frac{e^{\sqrt{A_i(1+\alpha)}x_i}}{\alpha_i^{2\sqrt{\frac{A_i}{1+\alpha}}}e^{-2\sqrt{A_i(1+\alpha)}x_i}-1}\\ &\quad\times\prod_{i=j+1}^{n-1}\cos\left(\sqrt{-\frac{A_i}{1+\alpha}}[-(1+\alpha)x_i+\beta_i]\right)\times[g_n(x_n)]^{\alpha+1}+a,\end{aligned}$$

where A,α,a,β_i,A_i *are real constants with* $\alpha<-1$, $A\neq 0$, $A_i<0$ *for* $i=1,\dots,j$, $A_i>0$ *for* $i=j+1,\dots,n-1$ *and* $\sum_{i=1}^{n-1}A_i=0$, *such that* $f>0$ *on* D.

(h)

$$\begin{aligned}f(x_1,\dots,x_n) &= \frac{A}{\alpha+1}\prod_{i=1}^{j}\left[(1+\alpha)x_i-\alpha_i\right]\\ &\quad\times\prod_{i=j+1}^{r}\left[\left(\alpha_i^{2\sqrt{\frac{A_i}{1+\alpha}}}e^{-2\sqrt{A_i(1+\alpha)}x_i}-1\right)e^{\sqrt{A_i(1+\alpha)}x_i}\right]\\ &\quad\times\prod_{i=r+1}^{n-1}\cos\left(\sqrt{-\frac{A_i}{1+\alpha}}[-(1+\alpha)x_i+\beta_i]\right)\times[g_n(x_n)]^{\alpha+1}+b,\end{aligned}$$

where $A,\alpha,a,\alpha_i,\beta_i,A_i$ *are real constants with* $\alpha\neq 0$, $\alpha>-1$, $A\neq 0$, $A_i>0$ *for* $i=j+1,\dots,r$, $A_i<0$ *for* $i=r+1,\dots,n-1$ *and* $\sum_{i=j+1}^{n-1}A_i=0$, *such that* $f>0$ *on* D.

(*i*)

$$f(x_1,\dots,x_n) = \frac{A}{\alpha+1}\prod_{i=1}^{j}\left[(1+\alpha)x_i-\alpha_i\right]\times\prod_{i=j+1}^{r}\frac{e^{\sqrt{A_i(1+\alpha)}x_i}}{\alpha_i^{2\sqrt{\frac{A_i}{1+\alpha}}}e^{-2\sqrt{A_i(1+\alpha)}x_i}-1}$$
$$\times\prod_{i=r+1}^{n-1}\cos\left(\sqrt{-\frac{A_i}{1+\alpha}}[-(1+\alpha)x_i+\beta_i]\right)\times[g_n(x_n)]^{\alpha+1}+b,$$

where $A,\alpha,a,\alpha_i,\beta_i,A_i$ *are real constants with* $\alpha<-1$, $A\neq 0$, $A_i<0$ *for* $i=j+1,\dots,r$, $A_i>0$ *for* $i=r+1,\dots,n-1$ *and* $\sum_{i=j+1}^{n-1}A_i=0$, *s.t.* $f>0$ *on* D.

Applying Theorem 18 for product production functions, we deduce the next result.

Corollary 4. [47] *Suppose* $D\subset\mathbb{R}^n$ *is an open domain and let* $f:D\to\mathbb{R}_+$ *be a product production function given by* $f(x_1,\dots,x_n)=\prod_{i=1}^{n}g_i(x_i)$, *where* $g_1,\dots,g_n$ *are twice differentiable functions. Then* f *has minimal isoquants iff, modulo a suitable translation,* f *reduces to the one of the next functions:*

(*a*)

$$f(x_1,\dots,x_n)=\alpha\prod_{i=1}^{n-1}(\beta_i x_i+\gamma_i)g_n(x_n),$$

where $\alpha,\beta_i,\gamma_i\in\mathbb{R}$ *with* $\alpha\neq 0$ *and* $\beta_i\neq 0$ *for* $i=1,\dots,n$, *such that* $f>0$ *on* D.

(*b*)

$$f(x_1,\dots,x_n) = \alpha\prod_{i=1}^{j}\left(C_{1i}e^{x_i\sqrt{A_i}}+C_{2i}e^{-x_i\sqrt{A_i}}\right)$$
$$\times\prod_{i=j+1}^{n-1}\left[C_{1i}\cos\left(x_i\sqrt{-A_i}\right)+C_{2i}\sin\left(x_i\sqrt{-A_i}\right)\right]\times g_n(x_n),$$

where j *is any number from the set* $\{1,\dots,n-2\}$, *while* $\alpha,A_i,C_{1i},C_{2i}\in\mathbb{R}$ *with* $A_1,\dots,A_j>0$, $A_{j+1},\dots,A_{n-1}<0$ *and* $\sum_{i=1}^{n-1}A_i=0$, *such that* $f>0$ *on* D.

(*c*)

$$f(x_1,\dots,x_n) = \alpha\prod_{i=1}^{r}(\beta_i x_i+\gamma_i)\times\prod_{i=r+1}^{s}\left(C_{1i}e^{x_i\sqrt{A_i}}+C_{2i}e^{-x_i\sqrt{A_i}}\right)$$
$$\times\prod_{i=s+1}^{n-1}\left[C_{1i}\cos\left(x_i\sqrt{-A_i}\right)+C_{2i}\sin\left(x_i\sqrt{-A_i}\right)\right]\times g_n(x_n),$$

where r *is any number from the set* $\{1,\dots,n-3\}$, s *is any number from the set* $\{r+1,\dots,n-2\}$, *while* $\alpha,\beta_i,\gamma_i,A_i,C_{1i},C_{2i}\in\mathbb{R}$ *with* $A_{r+1},\dots,A_s>0$, $A_{s+1},\dots,A_{n-1}<0$ *and* $\sum_{i=r+1}^{n-1}A_i=0$, *such that* $f>0$ *on* D.

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