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Fixed point results for some classes of mappings in geodesic spaces

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Abstract of the PhD Thesis

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Contents

Abstract

General objective. This study is motivated by the necessity of finding general classes of operators suitable for fixed point problems in geodesic spaces, along with versatile iteration procedures to make possible the numerical reckoning of solutions, once the existence is postulated. This Thesis aims to provide a consistent approach of this problem by means of two three-step iteration procedures, S_n iteration [35] and Thakur *et al.* iteration [37], both adapted to geodesic context. The general direction of study is at the boundary between nonlinear analysis & numerical algorithms, and the main results are related to fixed point problems, with nontrivial applications based on numerical modeling. The main conclusions resulting from the study of these iterative process are related to several features:

- *versatility.* Different classes of generalized nonexpansive operators are used in this Thesis: operators with property (E), L_2 operators, metric projections, operators with (P)-property. The iterative procedures utilized herein are adapted to curved geometric spaces: CAT(0) spaces and W-hyperbolic spaces.

- *instrumental value.* One main concern regarding the iteration procedure subject to analysis is related to the convergence of the sequence of iterations towards the solution of the studied problem. Most of the time, the results obtained refer to weak convergence. However, by introducing additional mild conditions for the underlying metric setting or for the operator involved in the iterative process, strong convergence results are also obtained.

- *qualitative aspects (stability and data dependence).* Iterative procedures are used in applied sciences to provide numerical algorithms for determining solutions to all sorts of nonlinear problems. The running of such algorithms is always subject to perturbations induced by the limitations of computer representation. Therefore, we have to constantly ensure that the approximations made during the running of the algorithm do not dramatically affect the estimation of the solution. A qualitative

analysis of an iterative process in general, and of the S_n process in particular, is motivated by such practical reasons.

Methodology. The methods used to perform the analysis of the involved iteration procedure are varied: the study of convergence relies on the uniqueness of the asymptotic center. Stability is analyzed according to the pattern provided by Harder and Hiks [20], and data dependence analysis uses the method initiated by Rus and Mureşan [34]. The comparative analysis regarding the efficiency of the studied process in relation to other procedures is performed using polynomiographic techniques, introduced by Kalantari [23]. Suitable coordinates are used to find the proper expression of the iterative process on the Poincaré half-plane. The algorithms used in numerical modeling for simulation are run in Matlab.

General state of art. The fixed point theory revolves around the fixed point equation $x = Tx$, associated to some self-mapping $T\colon X \to X$ acting on a nonempty set X on a metric space; for the pioneer source, see Caccioppoli [6]. One might say that there is actually an intense process going on, which encourages efforts to investigate and generalize Caccioppoli's results, through a continuous search for broader classes of operators and suitable metric frameworks, which ensure necessary topological and geometric properties for the existence of fixed points. To mention a few, we underline the outstanding contributions of Chatterjea [12], Ćirić [13], Garcia-Falset *et al.* [18], Gabeleh *et al.* [16, 17], Hardy and Rogers [21], Kannan [24], Reich [33], Suzuki [36].

Once the existence of fixed point is proven it is desirable to numerically compute it. That is why the study of multi-step iterative procedures, generally associated with generalized nonexpansive operators, was initiated within linear setting and developed recently both from the point of view of convergence behavior and from the perspective of stability and data independence. However, the transition to non-linear structures is not trivial, precisely because of the construction method of the iteration schemes. A careful look, however, highlights the fact that linearity is actually a too strong requirement, not necessarily essential, which can be replaced by certain convexity properties. For some pioneering results in this direction, please see the early works by Mann [29], Krasnosel'skii [26], Ishikawa [22], or more recent articles by Noor [30], Sintunavarat and Pitea [35], Thakur *et al.* [37]. Note that most of these remarkable results are initiated in normed linear space.

The metric alone is often not sufficient for studying specific practical issues, which allow to be modeled as fixed point problems. That is why the frameworks in which several original results were developed evolved towards enriched structures, such as the CAT(0) spaces [1, 19] or the W-hyperbolic spaces [25]. Both structures were identified as being proper settings for obtaining fixed point theoretical results as well, especially since many concepts valid in Banach spaces have exact counterparts in these geometric settings (see [14], [28]).

Thesis description: structure and content.

Chapter 1, **Operators with property $\mathbf{L_2}$ in CAT(0) spaces** ([7], [9]), deals with the solution of the common fixed point problem for two mappings belonging to the class of L_2 operators, which was very recently introduced in [27]. The formal definition of condition L_2 requires the self-mapping T to satisfy the inequality

$$\limsup_{n\to\infty} d(x_n, Tx) \leq \limsup_{n\to\infty} d(x_n, x),$$

for each almost fixed point sequence $\{x_n\}$ (shortly a.f.p.s, *i.e.*, a sequence $\{x_n\}$ such that $\{d(x_n, Tx_n)\}$ is convergent to zero).

The general framework used in this chapter is the CAT(0) setting, for which we were able to provide an example, Example 1, relying on the set of closed and bounded intervals of real numbers.

Example 1. Let $M = \{[a, b] \subset \mathbb{R} \mid a < b\}$ be a set of closed intervals in $\mathbb{R}$. On $M \times M$, we introduce the metric d given by the formula

$$d([a, b], [c, d]) = 2 \ln \frac{\sqrt{(d-b)^2 + (c-a)^2} + \sqrt{(d-a)^2 + (c-b)^2}}{\sqrt{2(b-a)(d-c)}},$$

for all $a, b, c, d \in \mathbb{R}$. Then, M is isometric with $\mathbb{H}^2$ (Poincaré half-plane), resulting that (M, d) is a CAT(0) space.

As instrument for approximating the solution of the common fixed point problem of a pair (F, G) of L_2 operators, we introduce a new iterative scheme, inspired by [35], properly adapted for the CAT(0) setting.

Algorithm 1. Let C be a nonempty and convex subset of a CAT(0) space (M, d). For two mappings $F\colon C \to C$ and $G\colon C \to C$ and $x_0 \in C$, the sequence $\{x_n\}$ is generated in three steps by the rules:

$$\begin{aligned} y_n &= (1-a_n)x_n \oplus a_n F x_n \\ z_n &= (1-b_n)x_n \oplus b_n G y_n \\ x_{n+1} &= (1-c_n)F z_n \oplus c_n F y_n, \quad n \geq 0, \end{aligned}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences bounded away from 0 and 1.

The goal in this chapter is to introduce suitable conditions such that the sequence $\{x_n\}$, generated by Algorithm 1 in relation to a pair (F, G) of L_2 mappings, to be strong convergent or Δ-convergent to a common fixed point $x \in \mathrm{Fix}(F, G)$ of the two operators.

In Lemma 1, we prove that a L_2 operator satisfies Browder's *demiclosedness principle*, where, instead of weak convergence, we assume Δ-convergence.

Lemma 1. *Let C be a subset of a complete $\mathrm{CAT}(0)$ space (M, d) and let $F : C \to M$ be a L_2 operator. If $\{x_n\} \subset C$ is an a.f.p.s. for F such that $x_n \overset{\Delta}{\to} x \in M$, then $Fx = x$.*

A set of lemmas are stated and proved next, providing some technical tools for the the main outcomes of the chapter. Lemma 2, Lemma 3 and Lemma 4 provide conclusions for an even more wider class of operators (quasi-nonexpansive mappings), and they refer to the solution set, and to some properties of the intermediate sequences of the iteration procedure.

Lemma 2. *Let $F : C \to C$ and $G : C \to C$, where C is a nonempty closed subset of a $\mathrm{CAT}(0)$ space, be two quasi-nonexpansive operators. Then the set $\mathrm{Fix}(F, G)$ is closed.*

Lemma 3. *Let (M, d) be a $\mathrm{CAT}(0)$ space and C be a nonempty and convex subset of M. Let $F : C \to C$ and $G : C \to C$ be two quasi-nonexpansive mappings such that $\mathrm{Fix}(F, G) \neq \emptyset$. Then, for the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, generated by Algorithm 1 and for any $q \in \mathrm{Fix}(F, G)$, the following limits*

$$\lim_{n\to\infty} d(x_n, q), \quad \lim_{n\to\infty} d(y_n, q), \quad \lim_{n\to\infty} d(z_n, q)$$

exist and are equal.

Lemma 4. *Let (M, d) be a complete $\mathrm{CAT}(0)$ space and C be a nonempty and convex subset of M. Consider $F : C \to C$ and $G : C \to C$ be two quasi-nonexpansive mappings which have at least one common fixed point and let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated by Algorithm 1. Then,*

(i) $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, z_n) = \lim_{n\to\infty} d(z_n, x_n) = 0$;

(ii) $\lim_{n\to\infty} d(x_n, Fx_n) = \lim_{n\to\infty} d(y_n, Gy_n) = 0$.

The structure of our iterative scheme makes it more difficult to establish whether $\{x_n\}$ is an almost fixed point sequence for the mapping G, than for the mapping F. We manage to avoid this obstacle by using the concept of equivalent sequences (two sequences $\{x_n\}$ and $\{y_n\}$ which satisfy $\lim_{n\to\infty} d(x_n, y_n) = 0$). Lemma 5 proves that two equivalent sequences have identical asymptotic centers and Δ-limits.

Lemma 5. *If $\{x_n\}$ and $\{y_n\}$ are two equivalent bounded sequences in a* CAT(0) *space (M, d), then $A(\{x_n\}) = A(\{y_n\})$. Moreover, if $x_n \overset{\Delta}{\to} p \in M$, then $y_n \overset{\Delta}{\to} p$ as well.*

Finally, Theorem 1 gives sufficient conditions such that the sequence $\{x_n\}$ generated by Algorithm 1 is Δ-convergent to a common fixed point of the two mappings F and G which satisfy condition L_2, provided that such points exist. An upgrade is settled in Theorem 2, where strong convergence is analyzed.

Theorem 1. *Let (M, d) be a complete* CAT(0) *space and C be a nonempty, closed and convex subset of M. If $F\colon C \to C$ and $G\colon C \to C$ are two mappings satisfying the condition* L_2 *such that* $\mathrm{Fix}(F, G) \neq \emptyset$*, then the sequence $\{x_n\}$, generated by Algorithm* 1*, is Δ-convergent to an element of* $\mathrm{Fix}(F, G)$.

Theorem 2. *Let (M, d) be a complete* CAT(0) *space and C be a nonempty, closed, convex subset of M and let $F\colon C \to C$ and $G\colon C \to C$ be two operators having the property* L_2*. Then, the iterative sequence $\{x_n\}$ converges to a point in* $\mathrm{Fix}(F, G)$ *if and only if* $\liminf_{n\to\infty} d(x_n, \mathrm{Fix}(F, G)) = 0$.

Further on, this chapter also includes an example, meant to illustrate to results described above. Let C be a point on the positive y-Axis of the Poincaré half-plane $\mathbb{H}^2$ and D be the disk centered at C with some fixed radius r. Consider the mappings $G(x, y) = (-x, y)$ and $FX = \frac{1}{2}C \oplus \frac{1}{2}X$, if $X = (x, y) \in \mathrm{int}D$, $FX = S_C\left(\frac{1}{2}C \oplus \frac{1}{2}X\right)$, if $X = (x, y) \in \partial D$, where $S_C(Y)$ denotes the symmetric of the point Y with respect to the point C. Both mappings F and G are proved to be L_2 operators. Moreover, the sequence induced by Algorithm 1 converges strongly to the point C.

Chapter 2, **A new approach to averaged mappings in CAT(0) spaces** ([7], preprint form [8]), considers new contractive and nonexpansivity conditions, inspired by the so-called enrichment technique introduced by Berinde and Păcurar [3, 5], which basically replaces the mapping T involved in some condition with its α-averaged version. This idea, applied for the setting of CAT(0) spaces and nonexpansive operators leads to Definition 1, involving the averaged mapping $T_\alpha x = (1 - \alpha)x \oplus \alpha T x$.

Definition 1. Let $T\colon X \to X$ be a given mapping in a CAT(0) space (X, d). If there exists $\alpha \in (0, 1]$ such that

$$d(T_\alpha x, T_\alpha y) \leq d(x, y),$$

for all $x, y \in X$, then T is called α-enriched nonexpansive mapping.

In the first part of the chapter, we develop the averaging technique for contractive conditions by allowing the averaging parameter α to vary and even take values outside of the unit interval. In geometric terms, this corresponds to points which are on the whole geodesic determined by the points x and Tx respectively. In order to do so, we assume that (X,d) is a CAT(0) space which has, additionally, the *geodesic extension property* and we denote by $S^{\lambda}_{[p,q]}$ the set of all points $r \in X$ such that there exists a geodesic line, containing the segment $[p,q]$, on which $r = (1-\lambda)p \oplus \lambda q$. In the degenerate case, when $p = q$, we shall take $S^{\lambda}_{[p,q]} = \{p\}$.

Based on the concept of selection function described in Definition 2, we define a more general enriched contractive condition in Definition 3. The motivation for this is that different pairs of points might require different averaging parameters in order to satisfy a contraction-type condition. Although the change does not seem significant, actually it is. For instance, Example 2 proves that our generalized contractions are not necessarily continuous (as in the case of enriched contractions in the sense of Berinde [5]). Moreover, some of them are not even quasi-nonexpansive mappings, as shown in Example 3.

Definition 2. Let (X,d) be a CAT(0) space. A function $B\colon X \rightrightarrows 2^{\mathbb{R}}$, that assigns to each $p \in X$ a subset $B(p) \subseteq \mathbb{R}$, with $B(p)\setminus\{0\} \neq \emptyset$, is called selection function.

Definition 3. Let $\gamma \in [0,1)$ and $B\colon X \rightrightarrows 2^{\mathbb{R}}$ be a selection function. A mapping $T\colon X \to X$ is said to be a (γ,B)-generalized contraction if, for every pair $(p,q) \in X \times X$, and every $\beta \in B(q)$, $\beta \neq 0$, there exists $\alpha \in B(p)$, $\alpha \neq 0$ such that

$$d(u,v) \leq \gamma d(p,q),$$

where $u \in S^{\alpha}_{[p,Tp]}$, $v \in S^{\beta}_{[q,Tq]}$.

It must be noted that, if $B(p) = \{\alpha\}$, $\forall p \in X$, for a given $\alpha \in (0,1]$, then we obtain the contractive version of Definition 1 and for $\alpha = 1$, one gets classical contractions.

Example 2. Let $X = \mathbb{R}$ be endowed with the Euclidean metric. Consider the mapping

$$T\colon X \to X, \qquad Tx = \begin{cases} 2x, & x \in \mathbb{Q}, \\ \dfrac{x}{2}, & x \in \mathbb{R}\setminus\mathbb{Q}, \end{cases}$$

and the selection function

$$B\colon X \rightrightarrows 2^{\mathbb{R}}, \quad B(x) = \begin{cases} \{-1\}, & \text{if } x \in \mathbb{Q}, \\ \{2\}, & \text{if } x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$$

Then, T is a (γ,B)-generalized contraction, for any $\gamma \in [0,1)$.

Example 3. Let X be the real plane $\mathbb{R}^2$, and $p = (x_1, y_1)$, $q = (x_2, y_2)$ two points. We endow X with the so-called jungle river metric

$$d(p,q) = \begin{cases} |y_1 - y_2|, & x_1 = x_2 \\ |y_1| + |x_1 - x_2| + |y_2|, & x_1 \neq x_2. \end{cases}$$

This is a well-known example of a $\mathbb{R}$-tree, and therefore a CAT(0) space, which additionally has the geodesic extension property.

Let now consider the mapping

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad T(x,y) = \begin{cases} (x, y+1), & y > 0 \\ (-x, y), & y = 0 \\ (x, y-1), & y < 0. \end{cases}$$

Clearly, the only fixed point of the mapping T is the origin $O(0,0)$. Also, it can be checked that T is not quasi-nonexpansive and therefore does not belong to many well-known classes of operators. Moreover, since T is not continuous, it is neither an enriched contraction in the sense of Berinde [3]. However, T is a (γ, B)-generalized contraction in the sense of Definition 3, for the selection function

$$B\colon X \rightrightarrows 2^{\mathbb{R}}, \quad B(x,y) = \begin{cases} \{-(|x| + |y|)\}, & \text{if } y \neq 0, \\ \left\{\dfrac{1}{2}\right\}, & \text{if } y = 0. \end{cases}$$

Regarding the numerical computation of fixed points, it is a common practice to construct those as limits of some iterates. In the setting of a CAT(0) space which has the geodesic extension property we use the Mann-type iteration process

$$\begin{cases} x_0 \in X \\ x_{n+1} \in S^{\alpha_n}_{[x_n, Tx_n]}, \quad \alpha_n \in B_\delta, \end{cases} \tag{1}$$

where B_δ is a bounded subset in $\mathbb{R}\backslash(-\delta, \delta)$, with $\delta > 0$ and T is a (γ, B)-generalized contraction. The main outcomes are included in Theorem 3 and they refer to the existence of fixed points and their numerical approach.

Theorem 3. *Let (X, d) be a complete* CAT(0) *space which has the geodesic extension property and let B_δ be a bounded subset in $\mathbb{R}\backslash(-\delta, \delta)$, for some $\delta > 0$. Consider a selection function $B\colon X \rightrightarrows 2^{B_\delta}$ (hence $B(x) \subset B_\delta$, for all $x \in X$). If $T\colon X \to X$ is a (γ, B)-generalized contraction, then T has a unique fixed point. Moreover, the iteration* (1) *converges to the unique fixed point of T, for a proper selection of step-sizes $\{\alpha_n\}$.*

The second part of this chapter is dedicated to a class of generalized nonexpansive mappings, as in Definition 4, whose description relies essentially on a given selection function.

Definition 4. Let (X,d) be a CAT(0) space and $B\colon X \rightrightarrows 2^{(0,1]}$ a selection function. A mapping $T\colon X \to X$ is called B-generalized nonexpansive if, for every pair $(p,q) \in X \times X$, and every $\beta \in B(q)$, there exists $\alpha \in B(p)$, such that

$$d(T_\alpha p, T_\beta q) \leq d(p,q).$$

Once more, we notice that, if $B(p) = \{\alpha\}$, $\forall p \in X$, for given $\alpha \in (0,1]$, we get the enriched nonexpansive mappings in the sense of Berinde whereas for $\alpha = 1$, we recover the nonexpansive mappings. Moreover, if B is a selection function such that $1 \in B(p)$, for all $p \in X$, then we recover the class of quasi-nonexpansive mappings.

The generality of the new operators is proved through Example 4, which provides a generalized nonexpansive mapping which fails to be quasi-nonexpansive.

Example 4. Let $X = [0,1] \times [0,1]$ together with the jungle river metric from the previous section and consider a mapping defined by

$$T\colon X \to X, \quad T(x,y) = \left(\frac{x+1}{2}, 1-x\right).$$

Then, T is a generalized nonexpansive mapping with respect to the selection function

$$B\colon X \rightrightarrows 2^{(0,1]}, \quad B((x,y)) = \begin{cases} \left\{\dfrac{y}{y+\frac{3}{2}(1-x)}\right\}, & \text{if } (x,y) \neq (1,0), \\ (0,1], & \text{if } (x,y) = (1,0), \end{cases}$$

where $\{\cdot\}$ denotes a singleton. On the other hand, T is not quasi-nonexpansive.

The main outcomes of this chapter refer to the existence of fixed points for generalized nonexpansive mappings, as well as to convergence results toward the solution, for a Mann type iteration scheme defined as follows:

$$x_{n+1} = T_{\alpha_n} x_n = (1-\alpha_n)x_n \oplus \alpha_n T x_n, \quad \alpha_n \in (0,1), \tag{2}$$

for a given $x_0 \in C$.

Theorem 4 establishes the existence of fixed points for a particular case of generalized nonexpansive mappings. More precisely, it considers a selection function $B\colon X \to (0,1]$, that is for each $x \in X$, $B(x)$ is a set consisting of a single element.

Theorem 4. *Let (X, d) be a complete* CAT(0) *space, C a bounded, closed and convex subset of X and $B\colon X \to (0, 1]$ a given selection function. Then, any B-generalized nonexpansive mapping $T\colon C \to C$ has a fixed point in C.*

Lemma 6 gives some necessary conditions under the assumption regarding the existence of fixed points, while Theorems 5 and 6 provide sufficient conditions for convergence.

Lemma 6. *Let (X, d) be a complete* CAT(0) *space, and C a bounded, closed and convex subset of X. Suppose that $T\colon C \to C$ is a B-generalized nonexpansive mapping, where $B\colon X \rightrightarrows 2^{[\delta,1]}$ is a given selection function. If x is a fixed point of T, then:*

i) *there exists a sequence $\{\alpha_n\} \subset (0, 1)$ such that the limit $\ell := \lim_{n\to\infty} d(x_n, x)$, where $\{x_n\}$ is generated by* (2)*, exists;*

ii) *moreover, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Theorem 5. *Let (X, d) be a complete* CAT(0) *space and let C be a bounded, closed and convex subset of X. If $T\colon C \to C$ is a B-generalized nonexpansive mapping for a given selection function $B\colon X \rightrightarrows 2^{[\delta,1]}$ with $F(T) \neq \emptyset$, then, for any $x_0 \in C$, there exists a sequence $\{\alpha_n\} \subset (0, 1)$ such that the sequence $\{x_n\}$ generated by* (2) *Δ-converges to a fixed point of mapping T.*

Theorem 6. *Let (X, d) be a complete* CAT(0) *space and let C be a bounded, closed and convex subset of X. If $T\colon C \to C$ is a B-generalized nonexpansive mapping for a given selection function $B\colon X \rightrightarrows 2^{[\delta,1]}$ with $F(T) \neq \emptyset$, which is also demicompact, then, for any $x_0 \in C$, there exists a sequence $\{\alpha_n\} \subset (0, 1)$ such that the iterative sequence $\{x_n\}$ generated by* (2) *converges to a fixed point of T.*

The last part of the chapter develops a calculation procedure for the points of a geodesic segment, in relation to the jungle river metric. The resulted formulas allowed us to produce algorithms for the construction of orbits specific to the Mann iteration.

In Chapter 3, **Fixed proximal pairs of E_r-mappings** ([7], [11]), the focus is pointed to the study of fixed proximal pairs of E_r-mappings in the setting of the CAT(0) spaces. A proximal fixed pair is any solution of the Problem 1, which was first formulated by Eldred *et al.* in [15] for relatively nonexpansive mappings in strictly convex Banach spaces and Hilbert spaces.

Problem 1. Let (X, Y) be a pair of two nonempty subsets of a metric space (M, d). Given a *noncyclic* mapping, *i.e.*, $T\colon X \cup Y \to X \cup Y$, such that $T(X) \subseteq X$ and $T(Y) \subseteq Y$, find $x \in X$ and $y \in Y$ such that $Tx = x$, $Ty = y$ and $d(x, y) = \operatorname{dist}(X, Y)$, where $\operatorname{dist}(X, Y) = \inf\{d(x, y)\colon x \in X, y \in Y\}$.

The goal of this chapter is to extend the study of Problem 1 to the setting of CAT(0) spaces, involving a more general class of mappings as in Definition 5, which we shall call noncyclic E_r-mappings (shortly, E_r-mappings), based on a condition similar to condition (E) of Garcia-Falset *et al.* [18]

Definition 5. Let (X, Y) be a pair of nonempty subsets of a metric space (M, d) and let (X_0, Y_0) denote the corresponding proximal pair. A noncyclic mapping $T\colon X \cup Y \to X \cup Y$ satisfies the noncyclic relative condition (E) (shortly (E_r)-condition) if $T(X_0) \subset X_0$, $T(Y_0) \subset Y_0$, and there exists $\mu \geq 1$ such that

$$d(x, Ty) \leq \mu d(x, Tx) + d(x, y) \quad \text{and} \quad d(y, Tx) \leq \mu d(y, Ty) + d(x, y),$$

for all $(x, y) \in X \times Y$.

A classification result is stated in Proposition 1, emphasizing the relation with quasi-noncyclic relatively nonexpansive mappings.

Proposition 1. Every mapping satisfying E_r-condition which has a best proximity pair is a quasi-noncyclic relatively nonexpansive mapping.

The main idea of this part is to study the convergence of the iterates generated by a Thakur-type iteration scheme [37] to the fixed proximal pairs of the new class of mappings. The iteration procedure, described in Algorithm 2, is properly adapted for the CAT(0) setting.

Algorithm 2. Let $x_1 \in X_0$ (or Y_0) and take $\{a_n\}, \{b_n\}$ two sequences in $[a, b]$ such that $0 < a \leq b < 1$ and define the iterative step as

$$\begin{aligned} z_n &= (1 - b_n)x_n \oplus b_n T x_n \\ y_n &= T((1 - a_n)x_n \oplus a_n z_n) \\ x_{n+1} &= T y_n, \end{aligned}$$

for all $n \geq 1$.

In the beginning, in Theorem 7, we establish the (P)-property for each pair of nonempty, closed and convex subsets in a CAT(0) space. The reasoning is based on some very specific properties of the metric projection relative to the proximal pair (X_0, Y_0). These properties are detailed in Lemma 7, Lemma 8, Lemma 9, Lemma 10.

Lemma 7. *Let (X,Y) be a pair of closed, convex sets in a complete* CAT(0) *space (M,d). If $x \in X_0$ and $y,z \in Y_0$ are three points such that $d(x,y) \leq d(x,z)$, then $d(x, P_X(y)) \leq d(x, P_X(z))$.*

Lemma 8. *Let (X,Y) be a pair of closed, convex sets in a complete* CAT(0) *space (M,d) and let $x,y \in X_0$. Then $d(x, P_Y(y)) = d(y, P_Y(x))$.*

Lemma 9. *Let (M,d) be a* CAT(0) *space and let X,Y be two nonempty, closed, convex subsets of M. Suppose additionally that Y is bounded. Then the subsets X_0 and Y_0 are nonempty, closed, convex and bounded.*

Lemma 10. *Let (X,Y) be a pair of nonempty, closed and convex subsets of a* CAT(0) *space (M,d) such that at least one of the sets X or Y is bounded. If*

$$P\colon X \cup Y \to X \cup Y, \quad P(x) = \begin{cases} P_X(x), & x \in Y \\ P_Y(x), & x \in X, \end{cases}$$

then:

i) $d(x, P(x)) = \mathrm{dist}(X,Y)$, *for any $x \in X_0 \cup Y_0$;*

ii) $P(X_0) \subseteq Y_0$ *and* $P(Y_0) \subseteq X_0$*;*

iii) *P is an isometry, that is, $d(P(x), P(\overline{x})) = d(x, \overline{x})$ and $d(P(y), P(\overline{y})) = d(y, \overline{y})$, for all $x, \overline{x} \in X_0$ and $y, \overline{y} \in Y_0$.*

Theorem 7. *Any pair (X,Y) of nonempty, closed and convex subsets in a* CAT(0) *space (M,d) has (P)-property.*

Another important property of the pair (X,Y), stated in Proposition 2, concerns the proximal Opial condition and provides the arguments for proving the demiclosedness-type result for E_r-mappings in Theorem 8.

Proposition 2. *Let (M,d) be a* CAT(0) *space and (X,Y) be a nonempty, closed, convex and proximal pair of M. Then, (X,Y) satisfies the proximal Opial condition.*

Theorem 8. *Let X and Y be two nonempty, bounded, closed and convex subsets of* CAT(0) *space (M,d). Assume that $T\colon X \cup Y \to X \cup Y$ is a mapping which satisfies the nonyclic E_r-condition and that the sequence $\{x_n\}$ Δ-converges to $x \in X$ such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then, $(x, P(x)) \in \mathrm{Prox}_{X\times Y}(T)$.*

Finally, important properties of the iterates generated by the Thakur-type iteration scheme are described in Lemma 11 and Lemma 12, such that to finally acquire the main outcomes which prove the Δ-convergence and strong convergence: Theorem 9 and Theorem 10.

Lemma 11. *Let X and Y be two nonempty subsets of a* CAT(0) *space (M, d) and let $T\colon X \cup Y \to X \cup Y$ be a mapping which satisfies the noncyclic E_r-condition. For arbitrary chosen $x_1 \in X_0$ consider the sequence $\{x_n\}$, generated by Algorithm* 2*. Then, for all $p \in F(T) \cap Y_0$ the limit $\lim\limits_{n\to\infty} d(x_n, p)$ exists. Furthermore, the sequence $\{x_n\}$ is bounded.*

Lemma 12. *Let (M, d) be a* CAT(0) *space and X, Y two nonempty subsets of M and let $T\colon X \cup Y \to X \cup Y$ be a mapping which satisfies the noncyclic E_r-condition. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by Algorithm* 2*. Then, $\lim\limits_{n\to\infty} d(x_n, z_n) = 0$ and $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$.*

Theorem 9. *Let (M, d) be a* CAT(0) *space and X, Y be two nonempty, bounded, closed and convex subsets of M. Suppose $T\colon X \cup Y \to X \cup Y$ is a mapping which satisfies the noncyclic E_r-condition and $\{x_n\}$ is a sequences generated by Algorithm* 2*. Then, the sequence $\{(x_n, P(x_n))\}$ Δ-converges to a fixed proximal pair of mapping T.*

Theorem 10. *Let (X, Y) be a nonempty, closed and convex pair of subsets in a* CAT(0) *space (M, d) such that at least one of the subsets is compact and let $T : X \cup Y \to X \cup Y$ be a E_r-mapping. Let $\{x_n\}$ be a sequence generated by Algorithm* 2*. Then, the sequence $\{(x_n, P(x_n))\}$ converges strongly to a fixed proximal pair of mapping T.*

Not least, several examples are meant to emphasize the practical value of the formal analysis revealed previously. Example 5 reveals a mapping which does not satisfy the condition (E), but satisfies E_r-condition. Example 6 underlines a mapping which satisfies E_r-condition, but is not noncyclic relatively nonexpansive.

Example 5. Consider the subsets in the Euclidean plane

$$X = \{a = (0, 1), b = (2, 1), c = (4, 1)\}, \quad Y = \{a' = (1, 0), b' = (3, 0), c' = (5, 1)\}$$

and the noncyclic mapping $T\colon X \cup Y \to X \cup Y$, given by the rule

$$T(a) = b,\ T(b) = a,\ T(c) = c,\ T(a') = a',\ T(b') = c',\ T(c') = b'.$$

This operator is not quasi-nonexpansive and hence does not satisfy the condition (E). On the other hand, it satisfies the noncyclic E_r-condition.

Example 6. Let $X = [0,1] \times \{1\}$ and $Y = [0,1] \times \{0\}$ be two subsets of $\mathbb{R}^2$ endowed with the usual Euclidean metric and let $T\colon X \cup Y \to X \cup Y$ be a mapping given by

$$T(x,1) = \left(\frac{x+1}{2}, 1\right) \text{ and } T(y,0) = \left(\frac{y+2}{3}, 0\right).$$

Then, T satisfies the noncyclic E_r-condition.

Chapter 4, **Operators with condition** (E) **in** W**-hyperbolic spaces** ([7], [10]), extends the underlying setting to more general hyperbolic frameworks. In [25], Kohlenbach used an additional convex structure to define a special class of metric spaces. It is precisely this key element that allows the study of iterative procedures to be extended beyond linear spaces. A new example, regarding closed and bounded real intervals is provided in Example 7, proving that this structure goes beyond the already known geometric models.

Example 7. Let $X = \{[a,b] : |a| < b\}$, be endowed with the metric d, given by the rule

$$d([a,b],[c,d]) = |(a+b) - (c+d)| + |(b^2 - a^2) - (d^2 - c^2)|,$$

for all $a, b, c, d \in \mathbb{R}$. Consider the mapping $W\colon X^2 \times [0,1] \to X$, by formula

$$W([a,b],[c,d],\alpha) = \frac{1}{2}\left[(1-\alpha)(a+b) + \alpha(c+d) - \frac{(1-\alpha)(b^2-a^2) + \alpha(d^2-c^2)}{(1-\alpha)(a+b) + \alpha(c+d)},\right.$$
$$\left.(1-\alpha)(a+b) + \alpha(c+d) + \frac{(1-\alpha)(b^2-a^2) + \alpha(d^2-c^2)}{(1-\alpha)(a+b) + \alpha(c+d)}\right]$$

where $\alpha \in [0,1]$. Then, the triple (X, d, W) is a hyperbolic space.

This chapter is carried out on a remarkable class of operators, that of mappings with condition (E), defined by Garcia-Falset *et al.* [18] and properly adapted in Definition 6 for hyperbolic setting.

Definition 6. Let (X, d, W) be a hyperbolic space. A mapping $T\colon S \to X$ satisfies condition (E_μ), for a given $\mu \geq 1$, provided that

$$d(x, Ty) \leq \mu d(x, Tx) + d(x, y),$$

for all $x, y \in S$, where S is a nonempty subset of the space X.

The classic iterations could be adapted to the hyperbolic setting, by using properly the convex structure. Hence, we explore through this chapter the three-step iteration procedure S_n, introduced by the authors Sintunavarat and Pitea [35] in 2016 and extended to the setting of hyperbolic spaces as follows:

Algorithm 3. Let $x_1 \in S$ and let the sequence $\{x_n\}$ be generated through the following iterative scheme:

$$\begin{cases} y_n = W(x_n, Tx_n, \beta_n) \\ z_n = W(x_n, y_n, \gamma_n) \\ x_{n+1} = W(Tz_n, Ty_n, \alpha_n), \end{cases}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0, 1)$.

Lemma 13 and Lemma 14 provide some original preparatory results, describing several features of the iterative sequence.

Lemma 13. *Let (X, d, W) denote a hyperbolic space, S a nonempty convex subset of X and $T\colon S \to S$ an operator satisfying $F(T) \neq \emptyset$ and $d(Tx, p) \leq d(x, p)$ for any $x \in X$, and $p \in F(T)$. If $p \in F(T)$ and the sequence $\{x_n\}$ is generated by Algorithm 3, then $\lim_{n\to\infty} d(x_n, p)$ exists.*

Lemma 14. *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity, S a nonempty convex subset of X and $T\colon S \to S$ an operator with condition (E) and with the property that $F(T) \neq \emptyset$. If the sequence $\{x_n\}$ is generated by Algorithm 3 with $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ bounded away from zero, then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Based on the properties above, the main results of this chapter provide a Δ-convergence result in Theorem 11 and some strong convergence outcomes in Theorem 12, Theorem 13 and Theorem 14.

Theorem 11. *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of convexity, S a closed nonempty convex subset of X and $T\colon S \to S$ an operator with condition (E) and with the property that $F(T) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by Algorithm 3, with $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ bounded away from zero, Δ-converges to a point $p \in F(T)$.*

Theorem 12. *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of convexity, S a compact nonempty convex subset of X and $T\colon S \to S$ an operator with condition (E) and with the property that $F(T) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by Algorithm 3 with $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ bounded away from zero, converges strongly to a point $p \in F(T)$.*

Theorem 13. *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of convexity, S a closed nonempty convex subset of X and $T: S \to S$ an operator with condition (E) and with the property that $F(T) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to a point $p \in F(T)$ if and only if $\lim_{n\to\infty} d(x_n, F(T)) = 0$.*

Theorem 14. *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of convexity, S a closed nonempty convex subset of X and $T: S \to S$ an operator with condition (E) and with the property that $F(T) \neq \emptyset$. If, in addition, the operator T also satisfies condition (A), then the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to a point of $F(T)$.*

In Chapter 5, **Qualitative study of S_n iteration in W-hyperbolic spaces** ([7], manuscript form [2]), the focus is on the qualitative study of the iterative procedure S_n in the W-hyperbolic setting, in connection to hyperbolic contractive mappings. The idea behind the concepts of stability and data independence comes from computer modeling of iterative algorithms. It actually answers the question: what happens to the the convergence pattern or to the real fixed point when working with a perturbed operator, or when numerical errors occur?

The stability of an iteration procedure means that the numerical errors that may occur during each iteration step do not interfere with the convergence behavior. This property was introduced by Harder and Hicks [20]. Later, Berinde [4], Olatinwo and Postolache [32] analyzed it in connection with various iteration procedures, especially in uniformly convex metric spaces. The original result concerning this specific issue, in connection to the S_n iteration procedure is included in Theorem 16, which is an outcome closely related to Theorem 15.

Theorem 15. *Let (X, d, W) be a complete hyperbolic space, S a closed nonempty convex subset of X and $T: S \to S$ a contraction mapping. Let $\{x_n\}$ be an iterative sequence generated by Algorithm 3, with $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$, satisfying the condition $\sum_{n=1}^{\infty} \alpha_n\beta_n\gamma_n = \infty$. Then, $\{x_n\}$ converges strongly to the unique fixed point of T.*

Theorem 16. *Let (X, d, W) be a complete hyperbolic space, S a closed nonempty convex subset of X and $T: S \to S$ a contractive mapping. Then, the iterative procedure S_n described in Algorithm 3, for $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ in $(0,1)$ with $\alpha_n\beta_n\gamma_n \geq a > 0$, $\forall n \geq 1$, is T-stable.*

On the other hand, fundamental results related to data independence were introduced over time by Rus and Mureşan [34], or Olatinwo [31]. It should be noted that the errors reach a minimum when the iteration depends only on the initial estimate and not on the operator itself (which may be subject to perturbations). The original outcome of this chapter concerning data independence is Theorem 17.

Theorem 17. *Let (X,d,W) be a complete hyperbolic space, S a closed nonempty convex subset of X and $T\colon S\to S$ a contraction with fixed point p. Let $\tilde{T}$ be an approximate mapping of the contraction mapping T with maximum admissible error ε. For a given initial estimate $x_1=\tilde{x}_1$, consider the iterative sequences $\{x_n\}$ and $\{\tilde{x}_n\}$ resulting through the iterative scheme S_n in Algorithm 3 applied in connection with operators T and $\tilde{T}$, respectively. Assume also that the real number sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy $s_n=\alpha_n\beta_n+\beta_n\gamma_n-\alpha_n\beta_n\gamma_n\geq\dfrac{1}{\lambda-\theta}$, for some $\lambda>\theta$. If $\lim\limits_{n\to\infty}\tilde{x}_n=\tilde{p}$, then $d(p,\tilde{p})\leq\dfrac{\lambda\varepsilon}{1-\theta}$.*

Additionally, an interesting example of contraction is provided in Example 8, together with an intuitive approach to the concept of approximate mapping.

Example 8. Let (X,d,W) be a complete W-hyperbolic space. If $p_0\in X$ and $\theta\in[0,1)$, then

$$T\colon X\to X,\ Tx=W(p_0,x,\theta)$$

is a contractive mapping with contraction constant θ and with unique fixed point p_0. Moreover, if $\varepsilon>0$ and $\tilde{p}_0\in\bar{B}\left(p_0,\dfrac{\varepsilon}{1-\theta}\right)$, then $\tilde{T}\colon X\to X$, $\tilde{T}x=W(\tilde{p}_0,x,\theta)$ is another θ-contraction, with fixed point $\tilde{p}_0$, which provides an approximate mapping of T, with maximum admissible error ε. Let us also point out that, since $d(p_0,\tilde{p}_0)\leq\dfrac{\varepsilon}{1-\theta}$, one has $d(p_0,\tilde{p}_0)\to 0$ as $\varepsilon\to 0$. This means that, any iteration scheme for which $x\to f(T,x)$ and $x\to f(\tilde{T},x)$ are convergent procedures toward the two fixed points, respectively, is data independent.

The chapter is completed with a computing procedure for the convexity map on the Poincaré disk model, which allows us further to perform numerical simulations and comparative analysis via polynomiography. We tested the resulted computational procedure on the contractive operator $Tx=W\left(p_0=\left(\dfrac{1}{2},-\dfrac{1}{2}\right),x,\theta=\dfrac{1}{2}\right)$ and initial estimate $x_1=\left(\dfrac{3}{5},\dfrac{3}{5}\right)$ and obtained the expected fixed point, via the S_n iteration procedure.

Finally, the chapter ends with a comparative analysis between the S_n procedure and the Picard iteration, using a visual method, polynomiography. Although introduced and used initially to determine the roots of complex polynomials (see [23]), polynomiography could be successfully applied to study efficiency of iterative algorithms. In this case, by applying the procedure for both hyperbolic S_n and Picard iterations, in connection with the contractive operator above, we reach to a clear conclusion: the S_n algorithm is more efficient than the Picard algorithm.

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