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Minimal and Maximal Operators Associated with some Classes of Pseudo-differential Operators and several Results on a Class of Nonlinear Equations

Summary of the PhD Dissertation

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Contents

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Summary of the PhD Dissertation

In the following, we present the structure of this thesis and the original results obtained within it.

In Chapter 1, **SG-Hypoelliptic Pseudo-differential Operators on $L^p(\mathbb{R}^n)$**, we study a subclass of a class of generalized SG-hypoelliptic pseudo-differential operators on $L^p(\mathbb{R}^n), 1 < p < \infty$, introduced by Camperi in [5]. We mention that this study was conducted in connection with the work of Viorel Catană [7].

The results in this chapter are included in the original paper [8], entitled ***Essential spectra and semigroups of perturbations of generalized SG-hypoelliptic pseudo-differential operators on $L^p(\mathbb{R}^n)$***, published in J. Pseudo-Differ. Oper. Appl. 13, 25 (2022), in which the author of this thesis is a co-author alongside Viorel Catană.

In **Section 1.1**, we will recall several important definitions and results related to the class of symbols $SG^m_{\rho,\delta}(\mathbb{R}^n)$, where $m = (m_1, m_2)$, $\rho = (\rho_1, \rho_2)$, $\delta = (\delta_1, \delta_2)$ are in $\mathbb{R}^2$, $0 \leq \delta_j < \rho_j \leq 1$, $j = 1, 2$, and we define SG-pseudo-differential operators with symbols in this class as follows (see [5]).

Definition 1. (see [5]) The class $SG^m_{\rho,\delta}(\mathbb{R}^n), m = (m_1, m_2), \rho = (\rho_1, \rho_2), \delta = (\delta_1, \delta_2)$ in $\mathbb{R}^2$, $0 \leq \delta_j < \rho_j \leq 1$, $j = 1, 2$, is defined as the set of all functions $\sigma(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ that satisfy the inequality

$$\left|\partial_\xi^\alpha \partial_x^\beta \sigma(x, \xi)\right| \leq C_{\alpha,\beta} \langle\xi\rangle^{m_1 - \rho_1|\alpha| + \delta_1|\beta|} \langle x\rangle^{m_2 - \rho_2|\beta| + \delta_2|\alpha|},$$

for all $\alpha, \beta \in \mathbb{N}^n$, $(x, \xi) \in \mathbb{R}^{2n}$, where $C_{\alpha,\beta}$ is a positive constant depending only on α, β, and $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$, $\langle x\rangle = (1 + |x|^2)^{1/2}$.

The elements of the class $SG^m_{\rho,\delta}(\mathbb{R}^n)$ are called global symbols of order m and type ρ, δ.

Certain classes of global symbols in the case where $\rho_1 = \rho_2 = 1, \delta_1 = \delta_2 = 0$ have been studied by Wong and Dasgupta in [12] and by Nicola and Rodino in [20]. In our thesis, we will consider the class of symbols with $\rho_1 = 1, \delta_2 = 0$, $0 < \rho_2 \leq 1$ and $0 \leq \delta_1 < 1$, thus we will be working in a more general framework.

If $m_j \leq m'_j, \rho'_j \leq \rho_j, \delta_j \leq \delta'_j, j = 1, 2$, then $SG^m_{\rho,\delta}(\mathbb{R}^n) \subseteq SG^{m'}_{\rho',\delta'}(\mathbb{R}^n)$.

We define

$$SG^{-\infty}(\mathbb{R}^n) = \bigcap_{m\in\mathbb{R}^2} SG^m_{\rho,\delta}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^{2n}),$$
$$SG^{\infty}(\mathbb{R}^n) = \bigcup_{m\in\mathbb{R}^2} SG^m_{\rho,\delta}(\mathbb{R}^n),$$

where $\mathcal{S}(\mathbb{R}^{2n})$ is the Schwartz space.

Let $\sigma \in SG^m_{\rho,\delta}(\mathbb{R}^n)$. We define the pseudo-differential operator $T_\sigma = \sigma(x, D)$ using the standard formula

$$(T_\sigma u)(x) = \sigma(x, D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x, \xi)\hat{u}(\xi)d\xi, \tag{1}$$

for all functions u in $\mathcal{S}(\mathbb{R}^n)$, where

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)dx, \xi \in \mathbb{R}^n.$$

We denote by $OPSG^m_{\rho,\delta}(\mathbb{R}^n)$ the space of all operators of the form (1) with symbols in $SG^m_{\rho,\delta}(\mathbb{R}^n)$.

Definition 2. (see [5]) Let $\sigma \in SG^m_{\rho,\delta}(\mathbb{R}^n)$ be a symbol. We say that σ is called a hypoelliptic symbol if there exist two positive constants $R, C_0, C_{0,\alpha,\beta}$ and $m = (m'_1, m'_2) \in \mathbb{R}^2$ such that

$$|\sigma(x, \xi)| \geq C_0\langle\xi\rangle^{m'_1}\langle x\rangle^{m'_2} \tag{2}$$

and

$$\left|D^\alpha_\xi D^\beta_x\sigma(x, \xi)\right| \leq C_{0,\alpha,\beta}|\sigma(x, \xi)|\langle\xi\rangle^{-\rho_1|\alpha|+\delta_1|\beta|}\langle x\rangle^{-\rho_2|\beta|+\delta_2|\alpha|}, \tag{3}$$

for every $\alpha, \beta \in \mathbb{N}^n$ and $(x, \xi) \in \mathbb{R}^{2n}$ with $|x| + |\xi| > R$.

We denote by $HSG^{m,m'}_{\rho,\delta}$ the class of hypoelliptic symbols of order (m, m').

Definition 3. (see [5]) Let $\sigma \in SG^m_{\rho,\delta}(\mathbb{R}^n)$ be a symbol. The symbol σ is called elliptic if it satisfies relation (2) for $m' = m$.

Remark 1. It is easy to observe that every elliptic symbol is hypoelliptic, since relation (2) for $m' = m$ implies (3).

In **Section 1.2**, we define the L^p-Sobolev spaces $H^{s_1,s_2,p}$ of order s_1, s_2, together with certain properties of these spaces, stating and later proving two results concerning the boundedness and the compactness of the SG-pseudo-differential operators (see

Theorems 1, 2). In the end of the Section 1.2, we state and prove two results involving the establishment of Sobolev-type estimates (see Theorem 3 and Corollary 1).

In the following, we present the definitions and results from this Section.

Let $s_1, s_2 \in \mathbb{R}$. We define the pseudo-differential operators $T_{\sigma_{s_1,s_2}}$, where

$$\sigma_{s_1,s_2}(x,\xi) = \langle x\rangle^{s_2}\langle \xi\rangle^{s_1} \in SG^{s_1,s_2}_{(1,1),(0,0)}\left(\mathbb{R}^n\right)$$

is an SG-elliptic symbol and $T_{\sigma_{s_1,s_2}} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is an isomorphism that can be extended to an isomorphism on $\mathcal{S}'(\mathbb{R})$ with inverse $T_{\sigma_{-s_1,-s_2}}$, where

$$\sigma_{-s_1,-s_2}(x,\xi) = \langle x\rangle^{-s_2}\langle \xi\rangle^{-s_1} \in SG^{s_1,s_2}_{(1,1),(0,0)}\left(\mathbb{R}^n\right).$$

Thus, if we denote symbolically $\sigma_{s_1,s_2}(x,D) = \langle x\rangle^{s_2}\langle D\rangle^{s_1}$, then $\sigma_{-s_1,-s_2}(x,D) = \langle D\rangle^{-s_1}\langle x\rangle^{-s_2}$.

Definition 4. For $1 < p < \infty, -\infty < s_1, s_2 < +\infty$, we define the L^p-Sobolev space $H^{s_1,s_2,p}$ of order s_1, s_2 as follows:

$$H^{s_1,s_2,p}\left(\mathbb{R}^n\right) = \left\{u \in \mathcal{S}'\left(\mathbb{R}^n\right); \langle x\rangle^{s_2}\langle D\rangle^{s_1}u \in L^p\left(\mathbb{R}^n\right)\right\}.$$

Then, $H^{s_1,s_2,p}\left(\mathbb{R}^n\right)$ is a Banach space where the norm is defined by the relation

$$\|u\|_{s_1,s_2,p} = \|\langle x\rangle^{s_2}\langle D\rangle^{s_1}u\|_{L^p(\mathbb{R}^n)}, \quad u \in H^{s_1,s_2,p}\left(\mathbb{R}^n\right),$$

where $\|\cdot\|_{L^p(\mathbb{R}^n)}$ is the norm in $L^p\left(\mathbb{R}^n\right)$.

We observe that $H^{0,0,p}\left(\mathbb{R}^n\right) = L^p\left(\mathbb{R}^n\right)$.

Moreover,

$$\bigcap_{(s_1,s_2)\in\mathbb{R}^2} H^{s_1,s_2,p}\left(\mathbb{R}^n\right) = \mathcal{S}\left(\mathbb{R}^n\right), \qquad \bigcup_{(s_1,s_2)\in\mathbb{R}^2} H^{s_1,s_2,p}\left(\mathbb{R}^n\right) = \mathcal{S}'\left(\mathbb{R}^n\right).$$

Theorem 1. *Let $\sigma \in SG^{(0,0)}_{(1,\rho_2),(\delta_1,0)}\left(\mathbb{R}^n\right), 0 < \rho_2 \leq 1, 0 \leq \delta_1 < 1$ be a symbol of global type and let $1 < p < \infty$.*

Then, $T_\sigma : L^p\left(\mathbb{R}^n\right) \to L^p\left(\mathbb{R}^n\right)$ is a bounded linear operator.

Theorem 2. *Let $\sigma \in SG^m_{\rho,\delta}, m = (m_1, m_2), \rho = (\rho_1, \rho_2), \delta = (\delta_1, \delta_2)$ in $\mathbb{R}^2$ such that $\rho_1 = 1, \delta_2 = 0, 0 < \rho_2 \leq 1, 0 \leq \delta_1 < 1$.*

Then, for $(s_1, s_2) \in \mathbb{R}^2$ and $1 < p < \infty$,$T_\sigma : H^{s_1,s_2,p}\left(\mathbb{R}^n\right) \to H^{s_1-m_1,s_2-m_2,p}\left(\mathbb{R}^n\right)$ is a bounded linear operator.

Moreover, the map $T_\sigma : H^{s_1,s_2,p}\left(\mathbb{R}^n\right) \to H^{t_1,t_2,p}\left(\mathbb{R}^n\right)$ is compact whenever $s_1 - t_1 > m_1, s_2 - t_2 > m_2$.

Theorem 3. *Let $T_\sigma \in OPSG^{m,m'}_{\rho,\delta}(\mathbb{R}^n), m' \leq m$ (i.e. $m'_j \leq m_j, j = 1,2,$ if $m = (m_1, m_2)$ and $m' = (m'_1, m'_2)$), be a SG-hypoelliptic pseudo-differential operator and assume $u \in \mathcal{S}'(\mathbb{R}^n), T_\sigma u \in H^{s,p}(\mathbb{R}^n), s = (s_1, s_2)$ in $\mathbb{R}^2, 1 < p < \infty$.*

Then, $u \in H^{s+m',p}(\mathbb{R}^n)$ and for every $t < s + m$ we have the estimate

$$\|u\|_{s+m',p} \leq C\left(\|T_\sigma u\|_{s,p} + \|u\|_{t,p}\right), \tag{4}$$

for a positive constant C depending on s.

Corollary 1. *Let $T_\sigma \in OPSG^{m,m'}_{\rho,\delta}(\mathbb{R}^n)$ be a SG-hypoelliptic pseudo-differential operator, where $m' = (m'_1, m'_2)$ is a couple of positive integers.*

Then,

$$\sum_{\substack{|\alpha| \leq m'_1 \\ |\beta| \leq m'_2}} \left\|x^\beta D^\alpha u\right\|_{0,p} \leq C\left(\|T_\sigma u\|_{0,p} + \|u\|_{0,p}\right),$$

where C is a positive constant.

At the beginning of **Section 1.3**, we associate to positive-order SG-pseudo -differential operators on $L^p(\mathbb{R}^n)$, where $1 < p < \infty$, the corresponding minimal and maximal operators.

Furthermore, we state and prove an analogue of the Agmon-Douglis-Nirenberg estimate from [1] for SG-hypoelliptic pseudo-differential operators (see Theorem 4).

Using this estimate, we prove a result regarding the inclusions between the domains of the minimal and maximal operators of SG-hypoelliptic pseudo-differential operators and other $L^p(\mathbb{R}^n)$-Sobolev spaces of order s_1, s_2, where $1 < p < \infty$, $-\infty < s_1, s_2 < \infty$ (Theorem 5).

Using a particular version of Erhling's inequality (see [23]), we prove some perturbation results concerning the Agmon-Douglis-Nirenberg inequality for SG-hypoelliptic pseudo-differential operators (see Theorems 6, 7, 8).

In addition, we state a conjecture regarding a version of Erhling's inequality (see Conjecture 1).

We now present the definitions and results from this Section.

Let $\sigma \in SG^m_{\rho,\delta}, m > 0$. We can consider the linear and continuous mapping $T_\sigma : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ as a linear operator $T_\sigma : \mathcal{D}(T_\sigma) \subset L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ with the dense domain $\mathcal{D}(T_\sigma) = \mathcal{S}(\mathbb{R}^n)$, where $1 < p < \infty$. It can be shown that this is a closed extension in $L^p(\mathbb{R}^n)$. We denote by $T_{\sigma,0}$ the smallest such extension and refer to it as the minimal operator of T_σ.

The domain $\mathcal{D}(T_{\sigma,0})$ of $T_{\sigma,0}$ consists of all functions u in $L^p(\mathbb{R}^n)$ for which there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $\mathcal{S}(\mathbb{R}^n)$ such that $\varphi_k \to u$ in $L^p(\mathbb{R}^n)$ and $T_\sigma\varphi_k \to f$ for some f

in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Moreover, if $u \in \mathcal{D}(T_{\sigma,0})$, it can be shown that the limit f does not depend on the choice of the sequence $\{\varphi_k\}_{k=1}^{\infty}$ in $\mathcal{S}(\mathbb{R}^n)$, and we can thus define $T_{\sigma,0}u = f$.

We consider the following operator:

$$T_{\sigma,1} : \mathcal{D}(T_{\sigma,1}) \subset L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad T_{\sigma,1}u = f, \quad u \in \mathcal{D}(T_{\sigma,1}),$$

and f in $L^p(\mathbb{R}^n)$ if and only if

$$(u, T_\sigma^* \varphi) = (f, \varphi), \quad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

where

$$(u, v) = \int_{\mathbb{R}^n} u(x)\bar{v}(x)\, dx,$$

for all measurable functions u and v on $\mathbb{R}^n$, assuming that the integral exists and T_σ^* is the formal adjoint of T_σ. It can be shown that $T_{\sigma,1}$ is a closed linear operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ with domain $\mathcal{D}(T_{\sigma,1})$ and $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{D}(T_{\sigma,1})$.

Furthermore, $T_{\sigma,1}u = T_\sigma u$, for all u in $\mathcal{D}(T_{\sigma,1}) = \{u \in L^p(\mathbb{R}^n) : T_\sigma u \in L^p(\mathbb{R}^n)\}$. We call $T_{\sigma,1}$ the maximal operator of T_σ. The operator $T_{\sigma,1}$ is the largest closed extension of T_σ, in the sense that if B is any closed extension of T_σ such that $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{D}(B^t)$, then $T_{\sigma,1}$ is an extension of B, where B^t is the true adjoint of B.

Theorem 4. *Let $\sigma \in HSG_{\rho,\delta}^{m,m'}(\mathbb{R}^n), 0 < m' \leq m$, be an SG-hypoelliptic symbol.*

Then, there exist positive constants C_1 and C_2 such that

$$C_1 \|u\|_{m',p} \leq \|T_\sigma u\|_{0,p} + \|u\|_{0,p} \leq C_2 \|u\|_{m,p}, u \in H^{m,p}(\mathbb{R}^n). \tag{5}$$

Remark 2. The estimate (5) is the analogue of the Agmon-Douglis-Nirenberg inequality from [1], applied to SG-hypoelliptic pseudo-differential operators. Therefore, we will refer to it as the SG-Agmon-Douglis-Nirenberg inequality.

Theorem 5. *Let $\sigma \in HSG_{\rho,\delta}^{m,m'}(\mathbb{R}^n), 0 < m' \leq m$, be an SG-hypoelliptic symbol.*

Then,

$$H^{m,p}(\mathbb{R}^n) \subseteq \mathcal{D}(T_{\sigma,0}) \subseteq \mathcal{D}(T_{\sigma,1}) \subseteq H^{m',p}(\mathbb{R}^n). \tag{6}$$

Conjecture 1. *Let $1 \leq p < \infty$ and $0 < s_j < t_j, j = 1, 2$. Then, for every positive number ε there exists a positive constant C_ε such that*

$$\|u\|_{s_1,s_2,p} \leq \varepsilon \|u\|_{t_1,t_2} + C_\varepsilon \|u\|_{0,0,p}, u \in H^{t_1,t_2,p}(\mathbb{R}^n).$$

Theorem 6. *Let* $\sigma \in HSG^{m,m'}_{\rho,\delta}, 0 < m' \le m, m = (m_1, m_2), m' = (m'_1, m'_2)$, *be an hypoelliptic symbol and let* V *be a measurable function on* $\mathbb{R}^n$ *such that there exists a positive constant* C *for which*

$$\|V\varphi\|_{0,p} \le C\|\varphi\|_{s,p}, \quad \varphi \in \mathcal{S}, \tag{7}$$

where $0 < s_1 < m'_1, s_2 \le 0, 1 < p < \infty$.

Then, there exist positive constants $\tilde{C}_1$ *and* $\tilde{C}_2$ *such that*

$$\tilde{C}_1\|\varphi\|_{m',p} \le \|(T_\sigma + V)\varphi\|_{0,p} + \|\varphi\|_{0,p} \le \tilde{C}_2\|\varphi\|_{m,p}, \quad \varphi \in \mathcal{S}. \tag{8}$$

Theorem 7. *Let* $\sigma \in HSG^{m,m'}_{\rho,\delta}, 0 < m' \le m, m = (m_1, m_2), m' = (m'_1, m'_2)$ *in* $\mathbb{R}^2$, *be an hypoelliptic symbol and let* V *be a measurable function on* $\mathbb{R}^n$ *such that the multiplication operator*

$$V : \mathcal{S}(\mathbb{R}^n) \subset \mathcal{D}(V) \subset L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n),$$
$$\mathcal{D}(V) = \{u \in L^p(\mathbb{R}^n); Vu \in L^p(\mathbb{R}^n)\},$$

satisfies the estimate

$$\|V\varphi\|_{0,p} \le C\|\varphi\|_{s_1,s_2,p}, \varphi \in \mathcal{S}(\mathbb{R}^n),$$

where $0 < s_1 < m'_1, s_2 \le 0, 1 < p < \infty$. *Then, there exist positive constants* C_1, C_2 *such that*

$$C_1\|u\|_{m',p} \le \|(T_\sigma + V)u\|_{0,p} + \|u\|_{0,p} \le C_2\|u\|_{m,p}, u \in H^{m,p}(\mathbb{R}^n).$$

Theorem 8. *Let* $\sigma \in HSG^{m,m'}_{\rho,\delta}(\mathbb{R}^n), 0 < m' \le m$, *be an SG-hypoelliptic symbol and let* V *be a SG-pseudo-differential operator* T_τ *with symbol* $\tau \in SG_{\rho,\delta}(\mathbb{R}^n), s < m'$ *(i.e.* $s_j \le m'_j, j = 1, 2$, *if* $s = (s_1, s_2)$ *and* $m' = (m'_1, m'_2)$*).*

Then, there exist positive constants C_1, C_2 *such that*

$$C_1\|u\|_{m',p} \le \|(T_\sigma + V)u\|_{0,p} + \|u\|_{0,p} \le C_2\|u\|_{m,p}, \quad u \in H^{m,p}(\mathbb{R}^n).$$

In **Section 1.4**, we state and prove some results concerning the essential spectrum of the perturbations of SG-hypoelliptic pseudo-differential operators with singular potentials on $L^p(\mathbb{R}^n), 1 < p < \infty$, introduced and studied in [21] (see Theorems 9, 10). A perturbation result for SG-hypoelliptic pseudo-differential operators with singular potentials on $L^p(\mathbb{R}^n)$ in the case where their symbol does not depend on x in $\mathbb{R}^n$ is also stated and proved (see Theorem 11). At the end of this section, we prove a result regarding strongly continuous contraction semigroups generated by SG-hypoelliptic pseudo-differential operators on $L^p(\mathbb{R}^n), 1 < p < \infty$ (see Theorem 12).

In the following, we present the theorems obtained in Section 1.4.

Theorem 9. *Let $\tau \in SG^{m}_{\rho,\delta}(\mathbb{R}^n), m = (m_1, m_2) \in \mathbb{R}^2$, be a global-type symbol, and let $s = (s_1, s_2) \in \mathbb{R}^2$ such that $m_j < s_j, j = 1, 2$, and let V be a measurable function on $\mathbb{R}^n$. Suppose that $M_{\alpha,p}(V) < \infty$ for some real α satisfying*

$$0 < \alpha < p\,(s_1 - m_1)\,,$$

where $1 < p < \infty$.

Then,

$$\|VT_\tau\varphi\|_{0,0,p} \le C\|\varphi\|_{s_1,s_2,p}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n)\,.$$

Thus, $H^{s_1,s_2,p}(\mathbb{R}^n) \subset \mathcal{D}(VT_\tau)$.

Moreover, if V satisfies the condition

$$\int_{|x-y|<1} |V(x)|^p dx \to 0 \text{ as } |y| \to \infty,$$

then $VT_{\tau,0}$ is a compact operator from $H^{s_1,s_2,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

Theorem 10. *Let $\sigma \in HSG^{m,m'}_{\rho,\delta}, 0 < m' \le m$, be a hypoelliptic symbol, let $\tau_j \in SG^{m_j}_{\rho,\delta}, 1 \le j \le r$, and let $V_j, 1 \le j \le r$, be measurable functions on $\mathbb{R}^n$. Suppose that*

$$V_j : \mathcal{D}(V_j) \subset L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

is the corresponding multiplication operator, and let $s^j = \left(s^j_1, s^j_2\right)$ in $\mathbb{R}^2$ such that $m^j < s^j < m'$, $m^j = \left(m^j_1, m^j_2\right)$, $m' = (m'_1, m'_2)$, and $s^j_2 \le 0$.

Suppose that

$$M_{\alpha_{j,p}}(V_j) < \infty, \quad 1 \le j \le r,$$

where

$$0 < \alpha_j < p\left(s^j_1 - m^j_1\right), \quad 1 \le j \le r,$$

and

$$\int_{|x-y|<1} |V_j(x)|^p \,dx \to 0 \text{ as } |y| \to \infty, \quad 1 \le j \le r, 1 < p < \infty.$$

Then, the operator

$$L : \mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)\,, L = T_\sigma + \sum_{j=1}^{r} V_j T_{\tau_j}$$

is closable in $L^p(\mathbb{R}^n)$. Moreover, if L_0 denotes its closure, then

$$H^{m,p}(\mathbb{R}^n) \subseteq \mathcal{D}(T_{\sigma,0}) = \mathcal{D}(L_0) \subseteq H^{m',p}(\mathbb{R}^n) \tag{9}$$

and $L_0 - T_{\sigma,0}$ is $T_{\sigma,0}$ compact.

Thus, $\sum_e (L_0) = \sum_e (T_{\sigma,0})$*, where we denote by* $\sum_e(T)$ *the essential spectrum of the operator* T *in the sense of Schechter. In particular, if* σ *is independent of* x *in* $\mathbb{R}^n$*, then* $\sum_e (L_0) = \sum_e (T_{\sigma,0}) = \sum (T_{\sigma,0}) = \{\sigma(\xi); \xi \in \mathbb{R}^n\}$*.*

Theorem 11. *Let* σ *be a real symbol as given in Theorem 10, which is independent of* x *in* $\mathbb{R}^n$*, and let* $P = \sum_{j=1}^{\tau} V_j T_{\tau_j}$ *be a symmetric operator that satisfies the hypotheses of Theorem 10 for* $p = 2$*, such that* $\tau_j, j = 1, 2, \ldots, r$*, is also independent of* x *in* $\mathbb{R}^n$*. Let* $L = T_\sigma + P$ *be an operator as in Theorem 10. Then,* L_0 *is a self-adjoint operator.*

Theorem 12. *Let* $\sigma \in HSG^{m,m'}_{\rho,\delta}(\mathbb{R}^n), m = (m_1, m_2), m' = (m'_1, m'_2), m_j \geq m'_j > 0, j = 1, 2$*, be an SG-hypoelliptic symbol such that* $T_{\sigma,0}$ *is the infinitesimal generator of a strongly continuous semigroup of contractions on* $L^p(\mathbb{R}^n), 1 < p < \infty, m'_1 > \alpha/p$*, where* α *is a positive constant.*

Let V *be a complex measurable function on* $\mathbb{R}^n$ *such that Re* $V(x) \leq 0$ *for almost all* $x \in \mathbb{R}^n$ *and* $M_{\alpha,p}(V) < \infty$*. Then,* $T_{\sigma,0} + V$ *is the infinitesimal generator of an one-parameter strongly continuous semigroup of contractions on* $L^p(\mathbb{R}^n)$*.*

In Chapter 2, **Minimal and Maximal Operators in an Abstract Framework**, starting from the article [22], we construct an abstract framework in which we study the minimal and maximal operators associated with an operator $A : \mathcal{D}(A) \subset X \to X$ with dense domain $\mathcal{D}(A)$, where X is a complex Banach space. In this abstract framework, we can recover Lebesgue or Sobolev spaces, together with various classes of pseudo-differential operators defined on these spaces, for example, M-hypoelliptic pseudo-differential operators (see [6], [7], [18]) or SG-pseudo-differential operators (see [5], [8], [12], [13], [19], [20]). Within our abstract framework, we also find the hybrid class of pseudo-differential operators from the paper [2].

We mention that the results in this chapter can be found in the original work entitled ***Some Properties of Minimal and Maximal Operators in an Abstract Framework***, published in U.P.B. Sci. Bull., Series A: Applied Mathematics and Physics, Vol. 87, Iss. 1 (2025), in which the author of this thesis is a co-author alongside Viorel Catană (see [11]).

In **Section 2.1**, we introduce the Bessel weighted potentials of order $(s_1, s_2) \in \mathbb{R}^2$ and, using them, we define the X-Sobolev spaces. Furthermore, we introduce a class of linear operators of order $(m_1, m_2) \in \mathbb{R}^2$ that are related to the X-Sobolev spaces, a class that could be viewed as an abstraction of the class of hybrid pseudo-differential operators from [2]. Additionally, we prove a supplementary property of the Bessel potentials, as well as of the spaces on which they are defined (see Proposition 1). Also, in this section, we recall some notations, definitions and results regarding the minimal and maximal operators associated with the operator A (see [22]).

In the following, we present a summary of the content of this section.

Let X be a complex Banach space with norm denoted by $\|\|_X$ and let S be a dense subspace of X. We assume that S is a topological space in which the topology is defined by a countable family of semi-norms $\{| \ |_j : j = 1, 2, \ldots\}$.

We say that a sequence $\{\varphi_k\}$ in S converges to an element φ in S if and only if $|\varphi_k - \varphi|_j \to 0$ as $k \to \infty$ for all $j = 1, 2, \ldots$.

We denote by S' the space of all linear and continuous functionals defined on the space S, and we denote by (u, φ) the value of a functional $u \in S'$ on an element $\varphi \in S$.

A functional u is continuous if and only if $(u, \varphi_k) \to 0$ as $k \to \infty$ for all sequences $\{\varphi_k\}$ converging to zero in S as $k \to \infty$.

We say that a sequence $\{u_k\}$ in S' converges to an element $u \in S'$ if and only if $(u_k, \varphi) \to (u, \varphi)$ as $k \to \infty$ for all $\varphi \in S$.

We assume that the spaces X and X' are continuously embedded in S'.

The definitions and notations used above are similar to those employed in distribution theory and are also used by Wong in [22].

Remark 3. We can particularize the spaces X and S to obtain nontrivial concrete cases. For example, let $X = L^p(\mathbb{R}^n), 1 \leq p < \infty$, let $S = S(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing functions, and let $S' = S'(\mathbb{R}^n)$ be the set of all tempered distributions.

Then, $X' = L^q(\mathbb{R}^n), \frac{1}{p} + \frac{1}{q} = 1$, and $S(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \subset S'(\mathbb{R}^n), 1 \leq p < \infty$.

In the following, we will present our abstract framework in which we will work. It should be mentioned that this abstract framework is similar to the one in the article [22].

We assume that there exists a family of reflexive Banach spaces $X^{\Lambda}_{s_1,s_2}$ in which the norms are denoted by $\|\|_{s_1,s_2,\Lambda,X}$, $-\infty < s_1, s_2 < \infty$, where $\Lambda : \mathbb{R}^n \to \mathbb{R}_+$ is a positive weight function, and also we assume that there exists a biparametric group of linear continuous applications $J^{\Lambda}_{s_1,s_2} : S' \to S', -\infty < s_1, s_2 < \infty$, which satisfy the following conditions:

(i) $J^{\Lambda}_{s_1,s_2}$ maps S into S, $-\infty < s_1, s_2 < \infty$, and $J^{\Lambda}_{\varepsilon,\varepsilon} : X \to X$ is a compact operator for every positive number ε.

(ii) $X^{\Lambda}_{s_1,s_2} = \left\{u \in S' : J^{\Lambda}_{-s_1,-s_2} u \in X\right\}, -\infty < s_1, s_2 < \infty$.

(iii)

$$\|u\|_{s_1,s_2,\Lambda,X} = \left\|J^{\Lambda}_{-s_1,-s_2} u\right\|_X, u \in X^{\Lambda}_{s_1,s_2}, -\infty < s_1, s_2 < \infty. \tag{10}$$

(iv)

Let $s_j \leq t_j$, $j = 1, 2$.

Then, $X^{\Lambda}_{t_1,t_2} \subseteq X^{\Lambda}_{s_1,s_2}$ and

$$\|u\|_{s_1,s_2,\Lambda,X} \leq \|u\|_{t_1,t_2,\Lambda,X}, \quad u \in X^{\Lambda}_{t_1,t_2}. \tag{11}$$

(v) $X^{\Lambda}_{s_1,s_2}$ can be continuously embedded in S', $-\infty < s_1, s_2 < \infty$.

(vi) S can be continuously embedded in $(X^{\Lambda}_{s_1,s_2})'$ and $(X^{\Lambda}_{s_1,s_2})'$ can be continuously embedded in S', $-\infty < s_1, s_2 < \infty$.

(vii)

$$(u, \varphi) = \overline{(\varphi, u)}, u \in X^{\Lambda}_{s_1,s_2}, \varphi \in S, -\infty < s_1, s_2 < \infty. \tag{12}$$

We denote by $J^{\Lambda}_{s_1,s_2}$ the weighted Bessel potentials of order $(s_1, s_2) \in \mathbb{R}^2$ and $X^{\Lambda}_{s_1,s_2}$ the X-Sobolev spaces of order $(s_1, s_2) \in \mathbb{R}^2$.

Definition 5. Let $T : S' \longrightarrow S'$ be a linear and continuous operator.

We assume that there exists a pair of real numbers (m_1, m_2) such that

$$T : X^{\Lambda}_{s_1,s_2} \to X^{\Lambda}_{s_1-m_1,s_2-m_2}$$

is a bounded linear operator for all $(s_1, s_2) \in \mathbb{R}^2$.

We say that T is an operator of order (m_1, m_2) if m_1 and m_2 are the smallest values for which $T : X^{\Lambda}_{s_1,s_2} \to X^{\Lambda}_{s_1-m_1,s_2-m_2}$ is a bounded linear operator.

If $m_1 = m_2 = -\infty$, we say that T is an infinitely smoothing operator.

Definition 6. Let $A : S \subset X \to X$ be a linear operator such that A maps S into S and its formal adjoint A^* maps S continuously into S.

We say that A is an operator of order (m_1, m_2) if the operator $A : S' \to S'$ is of order (m_1, m_2) (see relation (14) for the definition of the operator $A : S' \to S'$).

Remark 4. The biparametric family $X^{\Lambda}_{s_1,s_2}$, $s_1, s_2 \in \mathbb{R}$, of X-Sobolev spaces defines a biparametric abstract framework in which we can find, in certain particular cases, the theory of SG-pseudo-differential operators (see [5], [8], [12]) or the theory of a hybrid class of pseudo-differential operators (see [2]).

Proposition 1. *Let* $s_1, s_2, t_1, t_2 \in (-\infty, \infty)$.

Then,

i) $J^{\Lambda}_{t_1,t_2} : X^{\Lambda}_{s_1,s_2} \to X^{\Lambda}_{s_1+t_1,s_2+t_2}$ *is a unitary operator;*

ii) S *is dense in* $X^{\Lambda}_{s_1,s_2}$.

Remark 5. j) From (i), (ii) and Proposition 1, we obtain that $S \subset X \subset X^{\Lambda}_{0,0}$ and S is dense in $X^{\Lambda}_{0,0}$. Since S is dense in X, it follows that $X = X^{\Lambda}_{0,0}$.

jj) From Proposition 1 i), we obtain that $J^{\Lambda}_{t_1,t_2}$ is an operator of order $(-t_1, -t_2)$.

We will present some well-known definitions, notations and results related to the theory of minimal and maximal operators (see [2], [6], [7], [8], [22]).

Let X be a complex Banach space, S a dense subspace of X, and let A be a linear operator from X to X with domain S. We denote by X' the space of all linear and bounded functionals on X and by (x', x) the value of a functional x' in X' at an element x from X.

Definition 7. Let $\mathcal{D}(A^t)$ be the set of all functionals y' in X' for which there exists a functional x' in X' such that

$$(y', Ax) = (x', x), \quad x \in S. \tag{13}$$

It can be shown that for every y' in X', there exists at most one functional x' in X' that satisfies the relation (13).

Therefore, we can define $A^t y' = x'$, for all y' in $\mathcal{D}(A^t)$. A^t is called the true adjoint of A.

It can be proven that A^t is a closed linear operator from X' to X' with domain $\mathcal{D}(A^t)$. Additionally, we can observe that if B is a linear extension of A, then A^t is a linear extension of B^t.

Definition 8. Let A be a linear operator from X to X with domain S. The operator A is closable if and only if

$$\varphi_k \in S, \varphi_k \to 0 \text{ in } X, A\varphi_k \to x \text{ in } X \Rightarrow x = 0.$$

In the following, we will define the minimal operator associated with the operator A.

Assume that A is a closable operator. We can construct a closed linear extension A_0 of the operator A.

Definition 9. Let $\mathcal{D}(A_0)$ be the set of all elements x in X for which there exists a sequence $\{\varphi_k\}_{k=1}^{\infty}$ in S such that $\varphi_k \to x$ in X, and $A\varphi_k \to y$ for some y in X as $k \to \infty$.

We can define $A_0 x = y$, for all $x \in \mathcal{D}(A_0)$.

It can be proven that the definition of the operator A_0 does not depend on the particular choice of the sequence $\{\varphi_k\}_{k=1}^{\infty}$ and also it can be shown that A_0 is the smallest linear extension of A (i.e., if B is a linear extension of A, then B is a linear extension of A_0). A_0 is called the minimal operator associated with the operator A.

From now on, in this chapter, we will assume that X is a reflexive complex Banach space.

In order to define the maximal operator, we need to introduce the notion of formal adjoint. Assume that the space X and its dual X' can be continuously embedded into a topological space Y. Thus, the spaces X and X' can be viewed as subspaces of the space Y. Also, assume that there exists a subspace S of Y such that S is a dense subspace of both X and X'.

In the following definitions and results, we will assume that A is a linear operator from X to X with domain S.

Definition 10. The formal adjoint A^* of the operator A, if it exists, is defined as the restriction of the true adjoint A^t to the space S.

From Definition 10, we observe that the formal adjoint A^* exists if and only if S is contained in the domain of A^t.

Definition 11. We define the linear operator A_1 from X to X using the relation $A_1 = (A^*)^t$.

Let $\varphi \in S$. From Definition 11, we obtain that

$$(\varphi, A_1 x) = (A^* \varphi, x)$$

for all $x \in \mathcal{D}(A_1)$.

From the definition of the true adjoint, $\varphi \in \mathcal{D}(A_1^t)$ and $A_1^t \varphi = A^* \varphi$.

In the paper [22], it is shown that A_1 is a closed linear operator from X to X with domain $\mathcal{D}(A_1)$ containing the space S, the domain $\mathcal{D}(A_1^t)$ of the true adjoint of A_1 contains the space S, A_1 is a linear extension of A_0, and it is also proven that A_1 is the largest linear extension of A, with the property that the space S is contained in the domain of its adjoint (i.e., if B is a linear extension of A such that $S \subseteq \mathcal{D}(B^t)$, then A_1 is a linear extension of B). Thus, the operator A_1 is called the maximal operator associated with the operator A.

Let A be a linear operator from X to X with domain S. Assume that A maps S into S, and its formal adjoint A^* continuously maps S into S (i.e., if $\{\varphi_k\}$ is a sequence in S such that $\varphi_k \to 0$ in S as $k \to \infty$, then $A\varphi_k \to 0$ and $A^*\varphi_k \to 0$ in S as $k \to \infty$).

The linear operator A can be extended to the space S' as follows.

For all u in S', Au is an element in S' given by the relation

$$(Au, \varphi) = (u, A^* \varphi), \quad \varphi \in S. \tag{14}$$

It can be shown that $A : S' \to S'$ is a linear and continuous application.

In **Section 2.2**, we established some hypotheses in order to state and prove an analogue of the Agmon-Douglis-Nirenberg inequality for pseudo-differential operators in the case of the operator $A : \mathcal{D}(A) \subset X \to X$ with a dense domain $\mathcal{D}(A)$ (see Theorem 13). Using this inequality, we determined the domain of the minimal operator and proved that the minimal and maximal operators are equal when certain assumptions regarding the complex Banach space X and the operator A are satisfied (see Theorems 14, 15). Using the results obtained, we were able to state and prove a result regarding the existence and regularity of weak solutions of linear equations of the form $Au = f$ on the Banach space X (see Theorem 16).

Theorem 13. (Agmon-Douglis-Nirenberg inequality [1]) *Let A be a linear operator from X into X with domain S such that A maps S into S and its formal adjoint A^* maps S into S continuously. Suppose that A is of positive order (m_1, m_2) and there exists a linear operator B of order $(-m_1, -m_2)$ from X into X with domain S such that*

$$BA = I + R, \tag{15}$$

where I is the identity operator and R is an infinitely smoothing operator.

Then, there exist two positive constants C_1 and C_2 such that

$$C_1 \|x\|_{m_1,m_2,\Lambda} \leq \|Ax\|_{0,0,\Lambda} + \|x\|_{0,0,\Lambda} \leq C_2 \|x\|_{m_1,m_2,\Lambda}, \quad x \in X^{\Lambda}_{m_1,m_2}. \tag{16}$$

Theorem 14. *Let A be as in Theorem 13. Then, $\mathcal{D}(A_0) = X^{\Lambda}_{m_1,m_2}$.*

Theorem 15. *Let A be as in Theorem 13. Then, $A_0 = A_1$.*

Definition 12. Let $f \in X$. Then, an element u in X is called a weak solution of the linear equation $Au = f$ if $(A^*\varphi, u) = (\varphi, f)$, for all $\varphi \in S$.

Theorem 16. *Let $A : \mathcal{D}(A) \subset X \to X$ be a linear operator as in Theorem 13 and let $f \in X$. Then, every weak solution u of the linear equation $Au = f$ is in $X^{\Lambda}_{m_1,m_2}$.*

Remark 6. The previous theorem represents a regularity result in the sense that every weak solution u of the linear equation $Au = f$ belongs to a more regular space $X^{\Lambda}_{m_1,m_2}$ because $X^{\Lambda}_{m_1,m_2} \subset X^{\Lambda}_{0,0} = X$ by (11).

In **Section 2.3**, we proved two perturbation results. One of these refers to the Agmon-Douglis-Nirenberg inequality (see Theorems 17 and 18), while the other result refers to the strongly continuous semigroup of contractions generated by an operator A considered in Theorem 13 (see Theorem 19).

We will now present the results obtained in this section.

Theorem 17. *Let A be an operator as in Theorem 13, and let $V : \mathcal{D}(V) \subset X \to X$ with $S \subset \mathcal{D}(V)$ be a closed operator such that there exists a positive constant C for which*

$$\|V\varphi\|_{0,0,\Lambda} \leq C\|\varphi\|_{s_1,s_2,\Lambda}, \quad \varphi \in S, \tag{17}$$

where $0 < s_1 < m_1, s_2 \leq 0 < m_2$.

Then, there exist positive constants $\tilde{C}_1$ and $\tilde{C}_2$ such that

$$\tilde{C}_1\|\varphi\|_{m_1,m_2,\Lambda} \leq \|(A+V)\,\varphi\|_{0,0,\Lambda} + \|\varphi\|_{0,0,\Lambda} \leq \tilde{C}_2\|\varphi\|_{m_1,m_2,\Lambda}, \quad \varphi \in S. \tag{18}$$

Theorem 18. *Let A be an operator as in Theorem 13, and let $V : \mathcal{D}(V) \subset X \to X$ with $S \subset \mathcal{D}(V)$ be a closed operator that satisfies the estimate*

$$\|V\varphi\|_{0,0,\Lambda} \leq C\|\varphi\|_{s_1,s_2,\Lambda}, \varphi \in S, \tag{19}$$

where $0 < s_1 < m_1, s_2 \leq 0 < m_2$.

Then, there exist positive constants C_1, C_2 such that

$$C_1\|u\|_{m_1,m_2,\Lambda} \leq \|(A_0+V)\,u\|_{0,0,\Lambda} + \|u\|_{0,0,\Lambda} \leq C_2\|u\|_{m_1,m_2,\Lambda}, u \in X^{\Lambda}_{m_1,m_2}. \tag{20}$$

Theorem 19. *Let A be an operator as in Theorem 13 such that A is the infinitesimal generator of a strongly continuous semigroup of contractions on X.*

Let $V : \mathcal{D}(V) \subset X \to X$ be a maximal dissipative operator with $S \subset \mathcal{D}(V)$ such that

$$\|V\varphi\|_{0,0,\Lambda} \leq C\|\varphi\|_{s_1,s_2,\Lambda}, \varphi \in S, \tag{21}$$

where $0 < s_1 < m_1$, $s_2 \leq 0 < m_2$ and C is a positive constant.

Then, $A_0 + V$ is the infinitesimal generator of an one-parameter strongly continuous semigroup of contractions on X.

In **Section 2.4**, we stated and proved a theorem regarding X-Sobolev spaces (see Theorem 20), and using this theorem, we proved that the minimal operator associated with the operator A is Fredholm under certain hypotheses (see Theorem 21).

Theorem 20. *Let $s_1, s_2, t_1, t_2 \in \mathbb{R}$ such that $s_1 < t_1$ and $s_2 < t_2$.*

Then, the inclusion $i : X^{\Lambda}_{t_1,t_2} \hookrightarrow X^{\Lambda}_{s_1,s_2}$ is a compact operator.

Theorem 21. *Let $A : S \subset X \to X$ be an operator as in Theorem 13 such that it satisfies the equality $AB = I + L$, where B is the operator in Theorem 13 and L is an infinitely smoothing operator.*

Then, the bounded linear operator $A_0 : X^{\Lambda}_{m_1,m_2} \subset X \to X$ is Fredholm.

In Chapter 3, **On a Class of Generalized Nonlinear Equations Defined by Elliptic Symbols**, we introduce and study a class of nonlinear and nonlocal equations of infinite order of the type

$$f(-\Lambda(D))u = U(\cdot, u) \tag{22}$$

on $\mathbb{R}^n$, where $f : \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying certain ellipticity conditions, $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is a positive and continuous function such that $\Lambda(D)$ is a Fourier multiplier, where $D = (D_1, D_2, \ldots, D_n)\,, D_j = -i\frac{\partial}{\partial x_j}, 1 \leq j \leq n.$

The results obtained in Chapter 3 are included in the original work [10], entitled ***On a Generalized Class of Nonlinear Equations Defined by Elliptic Symbols***, published in Bull. Malays. Math. Sci. Soc. 47, 109 (2024), in which the author of this thesis is a co-author alongside Viorel Catană and Ioana-Maria Flondor.

The results obtained were inspired by the work of Bravo Vera [3] and the studies of Górka, Prado and Reyes (see [14], [15], [16], [17]). In these works, the authors considered a particular case, specifically that when the operator $f(\Delta)$ is defined using the Laplace operator on the Euclidean space $\mathbb{R}^n$ or on a Riemannian manifold (M, g). In these particular cases, results regarding the existence and regularity of solutions of the equations of the type

$$f(\Delta)u = U(\cdot, u) \tag{23}$$

were obtained.

We note that similar equations were studied in [3], [15], and [14].

In the work [4], Bravo, Prado and Reyes studied nonlinear pseudo-differential equations of the form

$$(a(-\Delta) + 1)^{s/2}u = U(\cdot, u), \tag{24}$$

where $s > 0$ and the symbols a belong to a certain class of symbols. Such equations were studied in the framework of Lebesgue spaces $L^p(\mathbb{R}^n)\,, 1 < p < \infty$, using Fourier multiplier theory. We observe that, in the case where $s = 2$ and $f(\Delta) = a(-\Delta) + 1$ in equation (24), we recover equation (23).

If we consider $\Lambda(\xi) = |\xi|^2, \xi \in \mathbb{R}^n$ (in this case $0 < \sigma \leq 1$), then $\Lambda(D) = -\Delta$ (where Δ is the Laplace operator on $\mathbb{R}^n$), and equation (27) becomes equation (1.4) studied in [3] and [17]. Therefore, in the particular case where $\Lambda(D) = -\Delta$, the results obtained in this chapter recover the results established by Górka, Prado and Reyes in [17], as well as those by Bravo Vera in [3].

Thus, it should be emphasized that, in this chapter of our thesis, we consider a more general case than the one studied in [3] and [17].

For example, in the particular case where $\Lambda(\xi) = |\xi|^3 + |\xi|^2, \xi \in \mathbb{R}^n$ (in this case $\frac{3}{4} < \sigma \leq 1$), then $\Lambda(D) = (-\Delta)^{\frac{3}{2}} - \Delta$.

To present some concrete examples of equations of the form (22), we can consider, for instance, $f : \mathbb{R} \to \mathbb{R}$, $f(s) = -se^{s^2}$.

In the case where $\Lambda(\xi) = |\xi| + |\xi|^2, \xi \in \mathbb{R}^n$, then $\Lambda(D) = (-\Delta)^{\frac{1}{2}} - \Delta$.

Hence, we obtain the equation

$$\left(\left(\Delta - (-\Delta)^{\frac{1}{2}}\right) e^{\left(\Delta - (-\Delta)^{\frac{1}{2}}\right)^2} + I\right) u = U(\cdot, u). \tag{25}$$

Another example of an equation of the form (22) can be obtained by considering the same form of the function f as in the previous example and $\Lambda(\xi) = |\xi|^2 + |\xi|^4, \xi \in \mathbb{R}^n$.

Then, $\Lambda(D) = -\Delta + \Delta^2$, and we obtain the equation

$$\left(\left(\Delta - \Delta^2\right) e^{\Delta^2 - 2\Delta^3 + \Delta^4} + I\right) u = U(\cdot, u). \tag{26}$$

In Chapter 3, we state and prove several results regarding the existence, uniqueness and regularity of solutions of the equations of the type (22), building upon the observations and techniques used in the works of the authors mentioned above.

In **Section 3.1**, we define the classes of symbols $\mathcal{G}^{\beta,\Lambda}$, $\beta > 0$, $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is a continuous and positive function with certain properties, as well as the Hilbert spaces $\mathcal{H}^{\beta,\Lambda}(f) \subset L^2(\mathbb{R}^n)$, where $\beta > 0$ and $f \in \mathcal{G}^{\beta,\Lambda}$, in which we will find the solutions of the equation (22). In addition, the formula for the operator $f(-\Lambda(D))$ is presented. Furthermore, within this section, we state and prove several results related to the classes of symbols $\mathcal{G}^{\beta,\Lambda}$ and the spaces $\mathcal{H}^{\beta,\Lambda}(f)$. One of these results refers to certain properties of the classes $\mathcal{G}^{\beta,\Lambda}$ (see Proposition 2), while two other results refer to the structure of the spaces $\mathcal{H}^{\beta,\Lambda}(f)$ (see Propositions 3 and 4).

In the following, we present the definitions and the results obtained in this Section.

As mentioned earlier, in this chapter we will study nonlinear equations of the form

$$f(-\Lambda(D))u(x) = U(x, u(x)), \quad x \in \mathbb{R}^n, \tag{27}$$

where u belongs to a subspace $\mathcal{H}^{\beta,\Lambda}(f) \subset L^2(\mathbb{R}^n)$ and

$$\Lambda(D)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \Lambda(\xi) \mathcal{F}u(\xi) d\xi = \mathcal{F}^{-1}(\Lambda(\xi)\mathcal{F}u(\xi))(x), \tag{28}$$

for all functions $u \in L^2(\mathbb{R}^n)$ such that $\Lambda(\cdot)\mathcal{F}u(\cdot) \in L^2(\mathbb{R}^n)$.

In the relation (28), we denote by $\mathcal{F}$ the Fourier transform and by $\mathcal{F}^{-1}$ the inverse Fourier transform.

It should be noted that the operator $\Lambda(D)$ is a Fourier multiplier, and the operator in (27) is, in fact, a function of Fourier multipliers.

Furthermore, $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is a continuous and positive function that satisfies the following condition:

$$\Lambda(\xi) \geq C_1 \left(1 + |\xi|^2\right)^{\sigma}, \quad |\xi| > R_1, \xi \in \mathbb{R}^n, \tag{29}$$

for some suitable real numbers C_1, R_1 and σ.

The measurable function $f : \mathbb{R} \to \mathbb{R}$ by means of which we can define the equation (27) will be called its *symbol*, and the function Λ above represents the *type* of the symbol f.

Therefore, in the following, we will define the class of symbols for which we will consider nonlinear equations of the form (27).

Definition 13. Let $\Lambda : \mathbb{R}^n \to \mathbb{R}$ be a positive and continuous function, and let β be a positive real number.

A measurable function $f : \mathbb{R} \to \mathbb{R}$ is called a symbol of order β and of type Λ if and only if the following two conditions are satisfied:

(i) $f(-s) \geq 0$ for all non-negative real numbers $s \in \mathbb{R}$;

(ii) There exist positive real numbers M, R such that

$$M(1 + \Lambda(\xi))^{\frac{\beta}{2}} \leq f(-\Lambda(\xi)), \quad |\xi| > R, \xi \in \mathbb{R}^n. \tag{30}$$

We denote by $\mathcal{G}^{\beta,\Lambda}$ the class of symbols of order β and of type Λ. A measurable function $f : \mathbb{R} \to \mathbb{R}$ that belongs to the space $\mathcal{G}^{\beta,\Lambda}$ will be called a $\mathcal{G}^{\beta,\Lambda}$-symbol, or simply a symbol.

In what follows, we will associate each $\mathcal{G}^{\beta,\Lambda}$-symbol f with a vector space $\mathcal{H}^{\beta,\Lambda}(f)$ as in the following definition.

Definition 14. Let f be a $\mathcal{G}^{\beta,\Lambda}$-symbol. A measurable real-valued function $g : \mathbb{R}^n \to \mathbb{R}$ belongs to the space $\mathcal{H}^{\beta,\Lambda}(f)$ if and only if the Fourier transform $\mathcal{F}(g)$ of the function g exists and satisfies

$$\int_{\mathbb{R}^n} [1 + f(-\Lambda(\xi))]^2 |\mathcal{F}g(\xi)|^2 \, d\xi < \infty. \tag{31}$$

We can endow the vector space $\mathcal{H}^{\beta,\Lambda}(f)$ with the following inner product:

$$(g_1, g_2)_{\mathcal{H}(\beta,\Lambda(f))} = \int_{\mathbb{R}^n} [1 + f(-\Lambda(\xi))]^2 \mathcal{F}g_1(\xi) \overline{\mathcal{F}g_2(\xi)} \, d\xi, \tag{32}$$

for all $g_1, g_2 \in \mathcal{H}^{\beta,\Lambda}(f)$.

Thus, the vector space $\mathcal{H}^{\beta,\Lambda}(f)$, endowed with the inner product defined above, becomes a Hilbert space.

We note that, in the case where $\Lambda(\xi) = |\xi|^2$, $\xi \in \mathbb{R}^n$, we obtain that $\mathcal{H}^{\beta,\Lambda}(f) = \mathcal{H}^{\beta}(f)$, where $\mathcal{H}^{\beta}(f)$ is the Hilbert space defined in [3] and [17].

Through a formal computation, it is convenient to define $f(-\Lambda(D))u$, for $u \in \mathcal{H}^{\beta,\Lambda}(f)$, as follows:

$$f(-\Lambda(D))u = \mathcal{F}^{-1}(f(-\Lambda(\xi))\mathcal{F}(u)(\xi)). \tag{33}$$

The operator

$$f(-\Lambda(D)) : \mathcal{H}^{\beta,\Lambda}(f) \to L^2(\mathbb{R}^n)$$

defined by relation (33), for any $u \in \mathcal{H}^{\beta,\Lambda}(f)$, is a well-defined linear operator.

Proposition 2. *The following assertions are true:*

(i) If $f \in \mathcal{G}^{\beta,\Lambda}$ and $g \in \mathcal{G}^{\gamma,\Lambda}$, then $f \cdot g \in \mathcal{G}^{\beta+\gamma,\Lambda}$, where β, γ are fixed positive real numbers and $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is a positive and continuous function which satisfies condition (29).

(ii) If $0 < \gamma < \beta$, then $\mathcal{G}^{\beta,\Lambda} \subset \mathcal{G}^{\gamma,\Lambda}$.

(iii) Let r be a real positive number and let $f_r : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f_r(s) = (1-s)^{\frac{1}{2}} - 1,$$

for all $s \in \mathbb{R}_-$, where $\mathbb{R}_-$ denotes the set of all non-positive real numbers.

Then, $f_r \in \mathcal{G}^{\beta,\Lambda}$, for all $\beta \le r$.

Proposition 3. *Let f be a $\mathcal{G}^{\beta,\Lambda}$-symbol. Then, the following statements are true:*

(i) For each $s \in \mathbb{R}$ such that $s \le \beta\sigma$, the embedding $\mathcal{H}^{\beta,\Lambda}(f) \hookrightarrow H^s(\mathbb{R}^n)$ holds.

(ii) For each integer number $k \ge 1$ such that $\frac{n}{2} + k < s \le \beta\sigma$, the embedding $\mathcal{H}^{\beta,\Lambda}(f) \hookrightarrow C^k(\mathbb{R}^n)$ holds.

(iii) For each $0 < \alpha < 1$ such that $\frac{n}{2} + \alpha = s \le \beta\sigma$, the embedding $\mathcal{H}^{\beta,\Lambda}(f) \hookrightarrow C^\alpha(\mathbb{R}^n)$ holds, where $C^\alpha(\mathbb{R}^n)$ is the space of Hölder continuous functions of order α.

Proposition 4. *A measurable function $g : \mathbb{R}^n \to \mathbb{R}$ belongs to the space $\mathcal{H}^{\beta,\Lambda}(f)$, $\beta\sigma > \frac{n}{2}$, if and only if $g = K_f^\Lambda * \tilde{g}$ for some $\tilde{g} \in L^2(\mathbb{R}^n)$.*

In addition, the Hilbert space $\mathcal{H}^{\beta,\Lambda}(f)$ endowed with the inner product given by (32) is isometric isomorphically with the Hilbert space $L^2(\mathbb{R}^n)$, and the transformation $\mathcal{K}_\Lambda : L^2(\mathbb{R}^n) \to \mathcal{H}^{\beta,\Lambda}(f)$ given by

$$\mathcal{K}_\Lambda(g) = K_f^\Lambda * g$$

is an isometric isomorphism, i.e., it is a bijection and

$$\left\|K_f^\Lambda * g\right\|_{\mathcal{H}^{\beta,\Lambda}(f)} = \|g\|_{L^2(\mathbb{R}^n)}.$$

In **Section 3.2**, we study the solutions of linear equations of the form

$$Lu = g, \quad g \in L^2\left(\mathbb{R}^n\right), \tag{34}$$

where the linear operator $L : \mathcal{H}^{\beta,\Lambda}(f) \to L^2\left(\mathbb{R}^n\right)$ is defined using the relation

$$L = f(-\Lambda(D)) + I. \tag{35}$$

In the previous relation, f is a $\mathcal{G}^{\beta,\Lambda}$-symbol, and $I : L^2\left(\mathbb{R}^n\right) \to L^2\left(\mathbb{R}^n\right)$ is the identity operator.

Furthermore, the linear operator L is defined by the relation

$$Lu = \mathcal{F}^{-1}([1 + f(-\Lambda(\xi))]\mathcal{F}u(\xi)), \quad u \in \mathcal{H}^{\beta,\Lambda}(f). \tag{36}$$

We state and prove an existence and uniqueness result for the solutions of equations of the form (34) (see Theorem 22) and, using this theorem, we prove under what conditions these equations admit real-analytic solutions (see Theorem 23). As an application of Theorem 23, we prove a real-analyticity and regularity result for the solutions of the linear generalized bosonic string equation (see Theorem 24). Moreover, we prove two results concerning the regularity and the invariance under rotations of the unique solutions of equations of the form (34) (see Propositions 5 and 6).

The original results obtained in this section are presented below.

Theorem 22. *Let f be a $\mathcal{G}^{\beta,\Lambda}$-symbol. Then, for each $g \in L^2\left(\mathbb{R}^n\right)$, there exists a unique solution $u_g \in \mathcal{H}^{\beta,\Lambda}(f)$ to the linear equation (34).*

Moreover, the equality

$$\|u_g\|_{\mathcal{H}^{\beta,\Lambda}(f)} = \|g\|_{L^2(\mathbb{R}^n)} \tag{37}$$

holds.

Theorem 23. *Let $f \in \bigcap_{\beta>0} \mathcal{G}^{\beta,\Lambda} = \mathcal{G}^{-\infty,\Lambda}$ (the equality is a notation) be a regularized symbol, where $\Lambda : \mathbb{R}^n \to \mathbb{R}$, the type of the symbol, is a positive and continuous radial function (i.e., $\Lambda(\xi) = \tilde{\Lambda}(|\xi|)$, for all $\xi \in \mathbb{R}^n$).*

In addition, suppose that there exists $c > 0$ such that

$$\lim_{m\to\infty} \frac{c^{2m}}{m!} \int_0^\infty \frac{r^{2m+n-1}}{[1 + f(-\tilde{\Lambda}(r))]^2} dr = 0. \tag{38}$$

Then, the unique solution $u_g : \mathbb{R}^n \to \mathbb{R}$ of the linear equation

$$(f(-\Lambda(D)) + I)u = g, \quad g \in L^2(\mathbb{R}^n), \tag{39}$$

is real-analytic, i.e., $u_g \in C^\omega(\mathbb{R}^n)$.

Theorem 24. *Let $\Lambda : \mathbb{R}^n \to \mathbb{R}$ be a positive continuous radial function (i.e., $\Lambda(\xi) = \tilde{\Lambda}(|\xi|)$ for all $\xi \in \mathbb{R}^n$) which satisfies (29) with $\sigma = 1$. Let $u_g : \mathbb{R}^n \to \mathbb{R}$ be the solution of the linear generalized bosonic string equation*

$$\left(\Lambda(D)e^{c\Lambda(D)} + I\right)u = g, \quad g \in L^2(\mathbb{R}^n), \tag{40}$$

where $c > 0$ is a positive constant.

Then, $u_g : \mathbb{R}^n \to \mathbb{R}$ is real-analytic in any ball $B(z, \tilde{\rho})$ in $\mathbb{R}^n$, with centre $z \in \mathbb{R}^n$ and the radius $\tilde{\rho} = \frac{\rho}{2\pi n^2}, 0 < \rho < \min(C_1, c)$, where C_1 is the positive constant in relation (29).

Remark 7. If we take $f : \mathbb{R} \to \mathbb{R}, f(s) = -se^{s^2}, s \in \mathbb{R}, \Lambda : \mathbb{R}^n \to \mathbb{R}$ which satisfies condition (29) with $\sigma = 1/2$ and such that $f \in \bigcap_{\beta>0} \mathcal{G}^{\beta,\Lambda} = \mathcal{G}^{-\infty,\Lambda}$ (i.e., f is a regularized symbol of type Λ), then it follows that f satisfies the hypotheses of Theorem 23.

For example, if $\Lambda(\xi) = |\xi|^2 + |\xi|, \xi \in \mathbb{R}^n$, then the solution u_g of the equation

$$\left(\left(\Delta - (-\Delta)^{\frac{1}{2}}\right)e^{\left(\Delta - (-\Delta)^{\frac{1}{2}}\right)^2} + I\right)u = g, \quad g \in L^2(\mathbb{R}^n),$$

is real-analytic in any ball $B(z, \tilde{\rho})$ with centre $z \in \mathbb{R}^n$ and the radius $\tilde{\rho} = \frac{\rho}{2\pi n^2}$, $0 < \rho < C_1$, where C_1 is the positive constant in (29).

Remark 8. Let $f : \mathbb{R} \to \mathbb{R}$ be a positive measurable function and let $\Lambda : \mathbb{R}^n \to \mathbb{R}$ be a positive continuous radial function (i.e., $\Lambda(\xi) = \tilde{\Lambda}(|\xi|), \xi \in \mathbb{R}^n$) which satisfies (29) such that

$$[1 + f(-\tilde{\Lambda}(r))]^2 \geq \tilde{\Lambda}(r)e^{c\tilde{\Lambda}(r)}, \tag{41}$$

for any $r \in \mathbb{R}^+$, where $c > 0$.

Then, the solution $u_g : \mathbb{R}^n \to \mathbb{R}$ of the linear equation

$$(f(-\Lambda(D)) + I)u = g, \quad g \in L^2(\mathbb{R}^n), \tag{42}$$

is real-analytic.

Proposition 5. *Let f be a $\mathcal{G}^{\beta,\Lambda}$-symbol and let $Lu = g$, $g \in L^2(\mathbb{R}^n)$ be the linear equation defined by the operator (36). Suppose, in addition, that $g \in \mathcal{H}^{\delta,\Lambda}(h)$ for some $\delta > 0$ and for some $h \in \mathcal{G}^{\delta,\Lambda}$.*

Then, the solution of this equation belongs to $\mathcal{H}^{\delta,\Lambda}(h) \cap \mathcal{H}^{\beta,\Lambda}(f) \cap \mathcal{H}^{\beta+\delta,\Lambda}(f \cdot h)$.

Proposition 6. *Let f be a $\mathcal{G}^{\beta,\Lambda}$-symbol and let $Lu = g$, $g \in L^2(\mathbb{R}^n)$, be the linear equation corresponding to this symbol. If g and Λ are invariant under rotations (i.e., for each rotation $R \in SO(n)$, $g(Rx) = g(x)$ and $\Lambda(R\xi) = \Lambda(\xi)$ for all $x, \xi \in \mathbb{R}^n$), then the solution u of the linear equation $Lu = g$, $g \in L^2(\mathbb{R}^n)$, is also invariant under rotations.*

In **Section 3.3**, we will consider a nonlinear form of equation (22). More precisely, we will study nonlinear equations of the type

$$f(-\Lambda(D))u(x) = U_\delta(x, u(x)), \quad x \in \mathbb{R}^n, \quad u \in \mathcal{H}^{\beta,\Lambda}(f), \tag{43}$$

where the right-hand side of equation (43) is defined by the relation

$$U_\delta(x, y) = -y + \delta V(x, y), \quad \text{for all } x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}. \tag{44}$$

In relation (44), $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a function with suitable properties and δ is a fixed nonnegative real number.

From (43) and (44), it follows that the considered nonlinear equation is equivalent to the equation

$$Lu(x) = \delta V(x, u(x)), \quad x \in \mathbb{R}^n, \tag{45}$$

where L is the operator defined in relation (35).

Using Banach's fixed-point theorem, we state and prove a result concerning the existence, uniqueness and regularity of solutions of equations of the type (43) (see Theorem 25). As a consequence of Theorem 25, we state and prove two additional results on uniqueness, real analyticity and regularity of the solutions of the nonlinear equations of the form (43) (see Corollaries 2 and 3). At the end of this section, we state and prove two results regarding the uniqueness, regularity and real analyticity of solutions of the considered nonlinear equations (see Theorem 26 and Corollary 4).

Below, we present the original results obtained in this section.

Theorem 25. *Let f be a $\mathcal{G}^{\beta,\Lambda}$-symbol and let $U_\delta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a function defined as in (44), where δ is a positive constant. Suppose, in addition, that $V(\cdot, 0) \in L^2(\mathbb{R}^n)$ and that there exists a function $h \in L^\infty(\mathbb{R}^n)$ such that the inequality*

$$|V(x, y_1) - V(x, y_2)| \leq |h(x)| \, |y_1 - y_2| \tag{46}$$

holds for all $x \in \mathbb{R}^n$ and $y_1, y_2 \in \mathbb{R}$.

Then, if $\delta > 0$ is sufficiently small, there exists a unique solution $u \in \mathcal{H}^{\beta,\Lambda}(f)$ to equation (43).

Corollary 2. *Let $f \in \bigcap_{\beta>0} \mathcal{G}^{\beta,\Lambda} = \mathcal{G}^{-\infty,\Lambda}$ be a regularized symbol, let $\Lambda : \mathbb{R}^n \to \mathbb{R}$ be a positive continuous radial function (i.e., $\Lambda(\xi) = \tilde{\Lambda}(|\xi|), \xi \in \mathbb{R}^n$) and let $U_\delta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \delta > 0$, be the function defined in (44) such that the hypothesis of Theorem 25 hold. Suppose, in addition, that the symbol $f : \mathbb{R} \to \mathbb{R}$ satisfies the hypotheses of Theorem 23.*

Then, there exists a real-analytic solution $u \in C^\omega(\mathbb{R}^n)$ of the equation (43).

Corollary 3. *Let f, β, V, and h be as in Theorem 25. Suppose, in addition, that there exists a positive real number r and an integer k with $\sigma(r+\beta) > n/2 + k$, such that*

$$V(\cdot, u) \in \mathcal{H}^{r,\Lambda}(f_r),$$

for all $u \in \mathcal{H}^{\beta,\Lambda}(f)$.

Then, the solution $\tilde{u}$ of the nonlinear equation (43) belongs to the class $C^k(\mathbb{R})$.

Consider the set

$$\mathcal{H}^{\beta,\Lambda}(f)_{\text{rad}} = \left\{ u \in \mathcal{H}^{\beta,\Lambda}(f) : \text{ for each rotation } R \in SO(n) \text{ we have } u(Rx) = u(x), \text{ for a.e. } x \in \mathbb{R}^n \right\}.$$

We observe that $\mathcal{H}^{\beta,\Lambda}(f)_{\text{rad}}$ is a closed subset of $\mathcal{H}^{\beta,\Lambda}(f)$.

Thus, $\mathcal{H}^{\beta,\Lambda}(f)_{\text{rad}}$ is a Hilbert space.

In particular, $\left(\mathcal{H}^{\beta,\Lambda}(f)_{\text{rad}}, \|\cdot\|_{\mathcal{H}^{\beta,\Lambda}(f)}\right)$ is a Banach space.

Theorem 26. *Let f, β, V and δ be as in Theorem 25, and let $\Lambda : \mathbb{R}^n \to \mathbb{R}$ be a positive continuous functions invariant under rotations.*

Suppose, in addition, that

$$U_\delta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R},$$

$$U_\delta(x, y) = -y + \delta V(x, y), \quad x \in \mathbb{R}^n, y \in \mathbb{R},$$

is invariant under rotation with respect to x.

Then, for $\delta > 0$ sufficiently small, there exists a unique solution $\tilde{u} \in \mathcal{H}^{\beta,\Lambda}(f)_{\text{rad}}$ to the nonlinear equation (43).

Corollary 4. *Let $f \in \bigcap_{\beta>0} \mathcal{G}^{\beta,\Lambda} = \mathcal{G}^{-\infty,\Lambda}$ be a regularized symbol, let $U_\delta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \delta > 0$, and $\Lambda : \mathbb{R}^n \to \mathbb{R}$ be two functions as in Corollary 2. Moreover, let us suppose that U_δ is invariant under rotation with respect to x.*

Then, there exists a unique real-analytic solution $u \in C^\omega(\mathbb{R}^n)$ of the equation (43).

At the end of our thesis, we presented an abstract of the original paper titled ***Generalized Fourier multipliers***, published in Ann. Funct. Anal. 14, 34 (2023) (see [9]), in which the author of this thesis contributed as a co-author alongside Viorel Catană and Ioana-Maria Flondor.

We note that the results obtained in the article [9], along with their proofs, are included in the PhD Dissertation authored by Ioana-Maria Flondor.

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